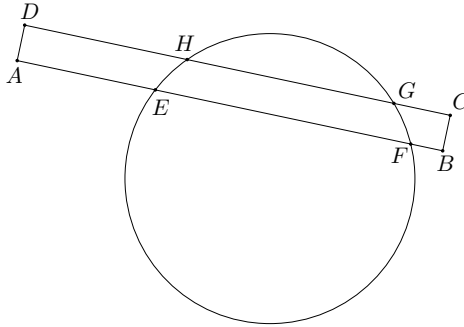


Problem 1 J. The numerator and denominator of Juan's fraction are positive integers whose sum is 2011. The value of this fraction is less than $\frac{1}{3}$. Find the greatest such fraction.

Problem 2 J. Rectangle $ABCD$ intersects a circle at points E, F, G, H as shown in the diagram. If $AE = 3$, $DH = 4$, and $GH = 5$, find EF .

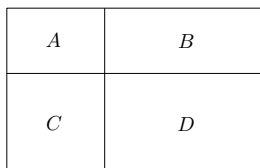


Problem 3 J. Find the sum of digits of the number $1 + 11 + 101 + 1001 + 10001 + \dots + \underbrace{10\dots01}_{50}$.

Problem 4 J. T-shirts were made in three colors, red, gray and blue. The number of red T-shirts is six smaller than the total number blue and gray T-shirts. Also, the number of gray T-shirts is ten smaller than the total number red and blue T-shirts. How many blue T-shirts were made?

Problem 5 J. There are 33 walnuts on the table in at least two piles. Each pile consists of at least 2 walnuts. After moving one walnut from each pile to the first one, all piles will have the same number of walnuts. What was the original number of piles? Find all possibilities.

Problem 6 J. A rectangle is divided by two segments parallel to its sides into four smaller rectangles. Let's label them A, B, C, D as in the diagram. Given that the perimeters of rectangles A, B, C are 2, 4, and 7, respectively, find all possible values of the perimeter of rectangle D .



Problem 7 J. Find pairwise distinct digits A, B, C such that

$$\begin{array}{r} A \\ AB \\ \underline{ABC} \\ BCB \end{array}$$

Problem 8 J. Find the area of a rectangle with perimeter 10 cm and diagonal $\sqrt{15}$ cm.

Problem 9 J. Andrei took N^3 equally sized white cubes and used them to form one big cube $N \times N \times N$. Then he colored its surface in red. Given that now one tenth of the total surface of the cubes is red, find N .

Problem 10 J. What is the least possible number of members of a math circle, where girls form more than 48.5% but less than 50% of the members?

Problem 11 J / 1 S. If you increase the number of this problem by n , you obtain the number of the most shocking problem. Whereas if you increase it by a two-digit number k , you obtain the number of the most playful problem. Moreover, we have $n^3 = k^2$. Find n and k , given that you have 30 problems left (including this one).

Problem 12 J / 2 S. Find positive integer n such that $6666^2 + 8888^2 = n^2$.

Problem 13 J / 3 S. Find the smallest positive integer, which ends with number 17, which is divisible by 17, and whose sum of digits is 17.

Problem 14 J / 4 S. Each pair of consecutive digits of a 2011-digit number is a multiple of either 17 or 23. Its last digit is 1. Find its first digit.

Problem 15 J / 5 S. A positive integer is called *awesome* if any other positive integer with the same sum of digits is greater. How many three-digit awesome numbers exist?

Problem 16 J / 6 S. Tim has found real numbers x, y, z satisfying $\frac{x-y}{z-y} = -10$. What are the possible values of $\frac{x-z}{y-z}$?

Problem 17 J / 7 S. The numbers $1, 2, \dots, 9$ are arranged in some order to form a nine-digit integer. Consider all triplets of consecutive digits and add the corresponding seven three-digit numbers. What is the largest result that can be obtained?

Problem 18 J / 8 S. A real number is written in each cell of a 10×10 square. Emily wrote down all products of two numbers from two distinct cells of the table and noticed that exactly 1000 of these products were negative. How many times did number 0 appear in the original square? Find all possibilities.

Problem 19 J / 9 S. Math kingdom started to produce a new set of coins. On the first day they created coins with value 1 MD (Math Dollar). Every other day they created coins with the smallest value which cannot be paid by at most ten existing coins. Which coins did they create on the 2011th day?

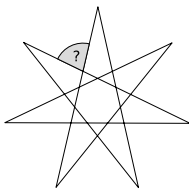
Problem 20 J / 10 S. Let the number p be the solution to this problem. Find the probability that a randomly chosen point inside a unit square is at least p units away from all sides.

Problem 21 J / 11 S. A 3×3 square is filled with integers such that the sums of the horizontal rows increase by two going downwards and the sums of the vertical columns double from left to right. Given that the sum of the numbers in one of the rows is 2011, find the sum of the numbers in the leftmost column.

Problem 22 J / 12 S. There are 2 boats, one on each side of a river bank. They both sail towards each other at a constant speed (not necessarily the same). The first time they meet, they

are 100 meters from one side of the bank. Once they reach the side of the bank, they turn around and move towards each other again. This time they meet 70 meters from the other side of the bank. How wide is the river?

Problem 23 J / 13 S. Vertices of a star form a regular heptagon. What is the magnitude of the marked angle?



Problem 24 J / 14 S. Find x such that $2^{2^{3^{2^2}}} = 4^{4^x}$.
Note: the order of operations is: $4^{3^2} = 4^9$.

Problem 25 J / 15 S. How many triplets of positive integers (a, b, c) exist, such that

$$\frac{\frac{a}{c} + \frac{a}{b} + 1}{\frac{b}{a} + \frac{b}{c} + 1} = 11$$

and $a + b + c \leq 30$?

Problem 26 J / 16 S. A circle ω with radius 1, center O and diameter AC is given in the plane. Draw a line ℓ through point O such that it is perpendicular to AC . Choose a point U on ℓ such that U is outside ω . Denote by B the second intersection of AU and ω and assume $BU = 1$. Find OU .

Problem 27 J / 17 S. Two nations A and B are in a battle with 1000 soldiers involved altogether. The armies take turns to attack. In each turn every living soldier from the attacking army shoots a soldier from the enemy's army. The battle ended (not necessarily by elimination of one of the sides) after three turns (A was shooting first, then B and finally A again). What is the least guaranteed number of survivors?

Problem 28 J / 18 S. All six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored in red. Each of the diagonals is either blue or red. Find the number of such colorings that each triangle $A_iA_jA_k$ ($i \neq j \neq k \neq i$) has at least one red side.

Problem 29 J / 19 S. Malcom told each Michal and Shri a positive integer. Further, he told them, that the numbers they heard were distinct and that their sum is a two-digit number. Then the following conversation took place:

Michal: "I cannot determine which one of us has the greater number."

Shri: "I can't determine it either, but I will tell you that my number is divisible by 17."

Michal: "Wow! Now, I can determine the sum of our numbers."

Find the value of this sum, given that the logic Michal and Shri used was flawless.

Problem 30 J / 20 S. There are 55 guests in a café, Turks and Indians. Each of them drinks either tea or coffee. An Indian speaks truth if he drinks tea and lies if he drinks coffee, whereas

with the Turks it is the other way round. For questions “Do you drink coffee?”, “Are you Turkish?” and “Is it raining outside?” the numbers of positive answers were 44, 33, and 22, respectively. How many Indians drink tea? Find all possibilities.

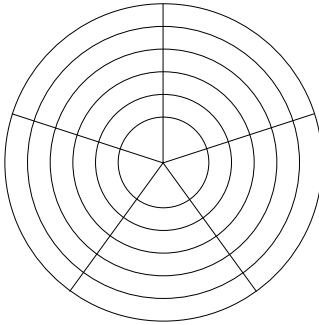
Problem 31 J / 21 S. Three digits were written to the end of a positive integer A . The resulting number was equal to the sum of numbers from 1 to A . Find all possible values of A .

Problem 32 J / 22 S. Alice, Betty, Claudia, Daniel, and Eli were playing doubles tournament in table-tennis. Each pair played against each other pair exactly once. Alice won 12 games and Betty won 6 games. How many games could Claudia win? Find all possibilities.

Problem 33 J / 23 S. Two players are playing a game on the given plan consisting of 30 cells. The rules are the following.

- players take turns,
- in one move a player colors one cell,
- in the first move, only a cell neighbouring with the edge can be colored. In any other moves, only a cell which is next to the last colored cell and is not further away from the center, can be colored,
- once a cell is colored, it cannot be colored again.
- the player who can no longer make a move, loses.

How many cells will be colored by the end of the game, in which both players play perfectly and the one who cannot win tries to make the game as long as possible?



Problem 34 J / 24 S. In triangle ABC with $AC = BC$, we find a point P ($P \neq B$) on the side AB , such that $PB < PA$ and $\angle ACP = 30^\circ$. Further, we find point Q such that $\angle CPQ = 78^\circ$ and points C and Q lie each on opposite side of AB . If the internal angles in triangles ABC and BQP have integral values (in degrees), find all possible values of angle BQP .

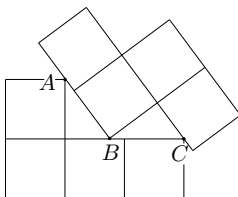
Problem 35 J / 25 S. Ten people in the theater are sitting next to each other in one row. After a break they sit in a new arrangement, so that only two people remained in their original positions and the remaining eight sat next to their former position. In how many ways could they have done that?

Problem 36 J / 26 S. A positive integer is written on each face of a cube. To each vertex we assign the product of the numbers written on the three faces intersecting at that vertex. The sum of the numbers assigned to the vertices is 165. What are the possible values of the sum of the numbers written on the faces?

Problem 37 J / 27 S. Two bicyclists are racing on a straight path at constant speeds. They both start at one end and everytime they hit the end of the path, they turn around and go the opposite direction. Eventually, they meet again at one of the endpoints. Before that, the slower bicyclist had traveled the path 35 times (in one of the directions) and the faster one 47 times. How many times did they meet head on?

Problem 38 J / 28 S. Find the largest positive integer such that all its digits (other than the first one and the last one) are smaller than the arithmetic mean of the two surrounding digits.

Problem 39 J / 29 S. Two tetrominoes made of 1×1 squares touch at points A, B, C as in the diagram. Find the distance AB .



Problem 40 J / 30 S. There are 100 points with integral coordinates given in the plane. We connect each pair with a segment. How many of these segments are guaranteed to have a midpoint with integral coordinates?

Problem 41 J / 31 S. A five-digit integer is called *irreducible* if it cannot be written as a product of two three-digit integers. What is the maximum possible number of consecutive irreducible integers?

Problem 42 J / 32 S. Real numbers x and y satisfy $(x + 5)^2 + (y - 12)^2 = 14^2$. Find the minimum possible value of $x^2 + y^2$.

Problem 43 J / 33 S. A sequence is defined as follows: $a_1 = 20$, $a_2 = 11$, and

$$a_{n+2} = a_n - \frac{1}{a_{n+1}},$$

as long as the right-hand is well-defined. Find the least t such that $a_t = 0$.

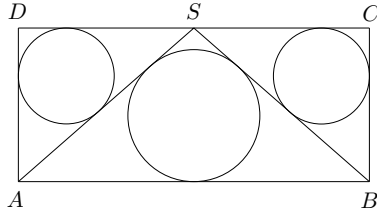
Problem 44 J / 34 S. Let ABC be an acute-angled triangle with altitudes AA' , BB' , CC' , intersecting at H . Given that

$$\frac{|AH|}{|HA'|} = 1, \quad \frac{|BH|}{|HB'|} = 2,$$

find $\frac{|CH|}{|HC'|}$.

Problem 45 J / 35 S. Every guest at a party (including Tim) knows exactly seven boys and ten girls there. What is the least possible number of people at the party?

Problem 46 J / 36 S. Let S be the midpoint of side CD of rectangle $ABCD$. The incircles of triangles ASD and BSC have radii both equal to 3 and the inradius of $\triangle ASB$ is 4. Find the sides of the rectangle.



Problem 47 J / 37 S. We write out all divisors of a positive integer n which are less than n from the greatest to the lowest. If n is the sum of the second and third divisor, we say that n is *additive*. How many additive numbers are there which are less than 15000?

Problem 48 J / 38 S. Find all real numbers x such that

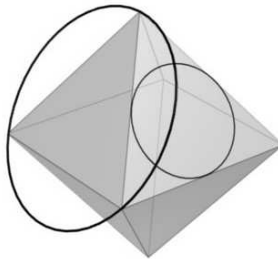
$$\frac{x - 49}{50} + \frac{x - 50}{49} = \frac{50}{x - 49} + \frac{49}{x - 50}.$$

Problem 49 J / 39 S. A position of a minute hand and an hour hand on the clock is called *valid*, if it might occur during one 12 hour cycle. Find the number of valid positions which remain valid after switching the two hands.

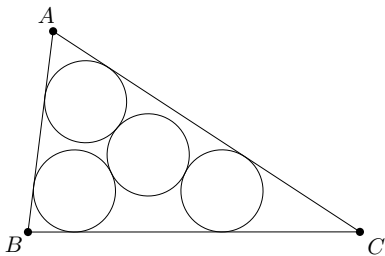
Problem 50 J / 40 S. Let a, b, c be nonzero real numbers, such that the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Find all possible real values of this root.

Problem 51 J / 41 S. Find all integers n such that both $16n + 9$ and $9n + 16$ are perfect squares.

Problem 52 J / 42 S. A regular octahedron with side length 2 is given in space. One circle is inscribed in one of the faces and another circle is circumscribed about an adjacent face. What is the minimal distance between the two circles?



Problem 53 J / 43 S. Let ABC be a triangle with circumradius 5 and inradius 2. Three circles with radius r are inscribed in angles BAC, CBA, ACB , respectively such that the circles are in the interior of the triangle and there exists another circle with radius r which is tangent to all three circles. Find r .



Problem 54 J / 44 S. Real numbers a, b, x, y satisfy

$$\begin{aligned}ax + by &= 3, \\ax^2 + by^2 &= 7, \\ax^3 + by^3 &= 16, \\ax^4 + by^4 &= 42.\end{aligned}$$

Find $ax^5 + by^5$.