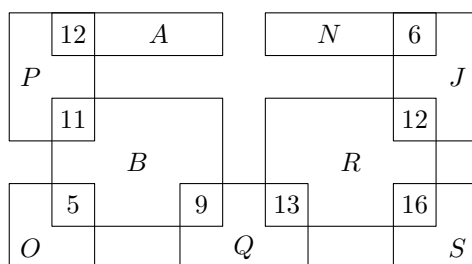


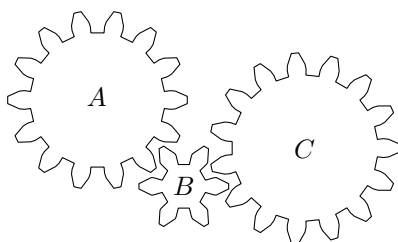
Problem 1. Let the letters in the rectangles be different non-zero digits. Each intersection of two rectangles contains the sum of the respective letters. Find the value of the five-digit number $NABOJ$.



Result. 14325

Solution. The sum 16 is only possible to obtain as $9 + 7$. Since $R = 9$ leads to the contradiction $J = N = 3$, we get $S = 9$ and $R = 7$. Step by step, we deduce the remaining values: $J = 5$, $N = 1$, $Q = 6$, $B = 3$, $P = 8$, $A = 4$ and $O = 2$. Therefore, we conclude $NABOJ = 14325$, which may be interpreted as the date of Náboj 2025.

Problem 2. How many complete rotations must gear C make for all the three gears to return to their original positions?



Result. 14

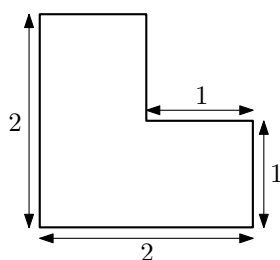
Solution. Gear A has 14 teeth, gear B has 6, and gear C has 15. Let us first find the smallest number of teeth by which the gear wheels must be turned so that all gear wheels return to their original position; this number must be a multiple of 14, 6, and 15. The least common multiple of these numbers is 210. Gear C must therefore be turned by 210 teeth and consequently must rotate $210/15 = 14$ times.

Problem 3. What is the largest ten-digit number in which every pair of identical digits has at least one smaller digit between them?

Result. 9897989698

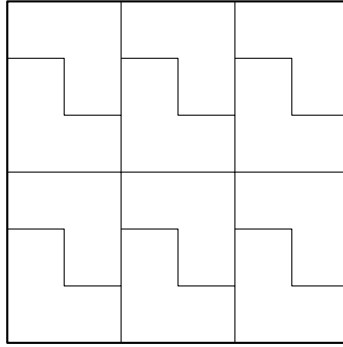
Solution. Let us look at the leftmost digit. We set it to 9, the largest possible value, and try to build the desired number by always adding the largest possible digit to the right so that the given condition is satisfied. Next, we cannot add 9, so we use 8 instead. Then we can add 9 again. Now, we cannot add 8 or 9, so the largest possible value is 7. Next, we can again put 9, 8, and 9 but then no digit from $\{9, 8, 7\}$. After placing 6, we can again add 9 and then, as the tenth digit, 8. This way we get the number $n = 9897989698$. We claim that this is the largest ten-digit admissible number. Indeed, let m be another one and look at the leftmost digit where m and n differ. Since our algorithm chose the largest available digit at this position we have $m \leq n$ regardless of the remaining digits.

Problem 4. What is the smallest sidelength of a square that can be completely covered without overlap using multiple copies of the right-angled L-shape shown below?

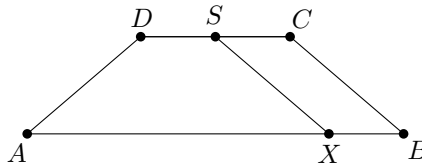


Result. 6

Solution. The area of the square must be a multiple of 3, the area of the triomino, and it is easy to see that a 3×3 square cannot be fully covered. However, a 6×6 square can be covered, and one possible arrangement is shown below.



Problem 5. In an isosceles trapezoid $ABCD$ with bases AB and CD , the sidelengths satisfy $BC = CD = AD$. Additionally, let S be the midpoint of DC and X a point on AB such that XS is parallel to BC . Given that the perimeter of $ABCD$ is 50 and the perimeter of $AXSD$ is 38, find the perimeter of the parallelogram $XBCS$.



Result. 36

Solution. The difference of perimeters is exactly $XB + CS = 2 \cdot CS = CD$, which further equals BC and XS , so the result is $3 \cdot (50 - 38) = 36$.

Problem 6. Elaine selected a two-digit number, with no zeros, and multiplied it by its reverse. The result was a four-digit number that started with 3 and ended with 7. What was the larger of the two numbers she multiplied?

Result. 93

Solution. Let x, y be the two digits of the number. Observe that the units digit of Elaine's product is the same as the units digit of the product $x \cdot y$. There are only two ways to multiply two digits and get a number ending with 7, namely $1 \cdot 7 = 7$ and $3 \cdot 9 = 27$. We can easily rule out the option $17 \cdot 71$ because this product is too small, so the correct option is $39 \cdot 93 = 3627$ and the answer is 93.

Problem 7. Kurt is playing a card game with a standard 52-card deck (13 ranks of the 4 suits). In each turn, a player can either draw a card or play a card from their hand that shares either the same rank or the same suit as the card on top of the playing pile. In the previous rounds, Kurt was very unlucky and ended up drawing many cards, which got him wondering: What is the minimum number of cards N that he must have in his hand such that, no matter which N cards he holds and regardless of the card on top of the playing pile, he is guaranteed to be able to play at least one card?

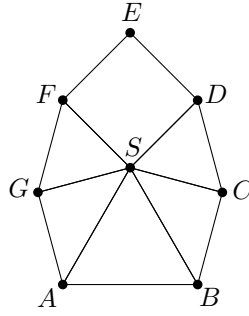
Result. 37

Solution. If Kurt holds all combinations of three suits and twelve ranks (a total of $3 \cdot 12 = 36$ cards), then it is possible that the card on top of the playing pile has the missing fourth suit and the missing thirteenth rank. In this case, none of Kurt's cards would be playable, therefore N is at least 37.

On the other hand, the card on the top of the pile shares a suit with 12 cards and a rank with 3 cards. Since the total number of cards is 52, and at least one of them is in the pile, the maximum number of unplayable cards in Kurt's hand is $52 - 1 - 12 - 3 = 36$, so holding 37 cards, he will certainly be able to play at least one of them.

Problem 8. Heptagon $ABCDEFG$ is composed of six polygons sharing a common vertex S : three equilateral triangles (ABS , CDS , FGS), two isosceles right triangles (BCS , GAS with right angle at C , G , respectively), and a

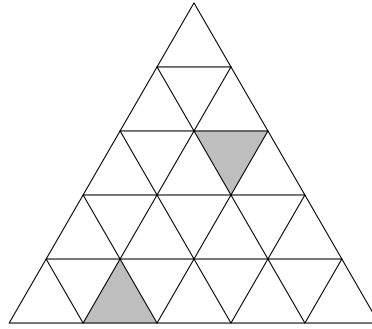
square ($DEFS$). Find the measure of angle SAE in degrees.



Result. 15°

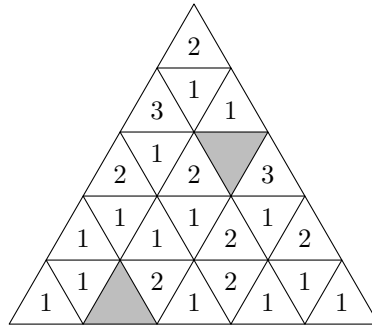
Solution. Since triangle FGS is equilateral, we get $FS = GS$. Therefore, the isosceles right-angled triangles GAS and FSE are congruent, which implies $ES = AS$. Hence, triangle EAS is isosceles. Using $\angle ESA = 45^\circ + 60^\circ + 45^\circ = 150^\circ$, we obtain $\angle SAE = \frac{1}{2}(180^\circ - \angle ESA) = 15^\circ$.

Problem 9. Consider the given triangular grid, where two tiles are shaded in grey. How many triangles can be drawn along the grid lines such that they do not contain either of the grey tiles?



Result. 34

Solution. Let us write into each tile how many permissible triangles have this particular tile as their top or bottom corner (depending on the orientation of the tile):



The sought number is simply the sum of all these values.

Problem 10. We call a four-digit number *intriguing* if it has the following property: when its hundreds digit is removed, the resulting three-digit number is one-ninth of the original four-digit number. For example, the number 2025 is intriguing, since $225 = \frac{1}{9} \cdot 2025$. Find the largest intriguing four-digit number.

Result. 6075

Solution. Let $N = \overline{abcd}$ be an intriguing number and put $n = \overline{cd}$. Then $N = 1000a + 100b + n$ and when we remove the hundreds digit, we obtain a number $M = 100a + n$. Multiplying the equality $M = \frac{1}{9}N$ by 9, we get

$$9(100a + n) = 1000a + 100b + n$$

and further rearranging and dividing by 4 yields

$$25(a + b) = 2n.$$

From this equality we infer that $a + b$ has to be an even number and less than $\frac{2 \cdot 100}{25} = 8$, since $n < 100$. This implies that $a + b$ is at most 6 and in order to maximize the number N , we pick $a = 6$ and $b = 0$, leading to $n = 75$. It is easy to check that $N = 6075$ satisfies the condition from the statement.

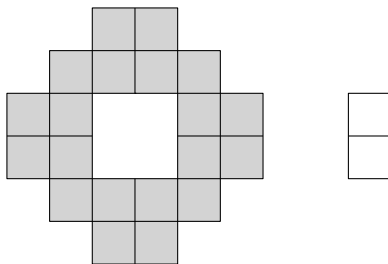
Problem 11. A cargo ship is designed to carry three types of liquids simultaneously: ethanol, oil, and mercury. Each liquid has its own maximum capacity: 10 tons of ethanol, 30 tons of oil, and 60 tons of mercury. On its journey from Prague to Hamburg, the ship is loaded with a total of 85 tons of freight, consisting of these liquids. On the return trip, the ship carries the same amount of ethanol, twice the amount of oil, and one-third the amount of mercury compared to the first journey. How much freight, in tons, does the ship carry on its return trip?

Result. 60

Solution. Since the ship was loaded with 85 tons on the first journey, it had to carry at least 15 tons of oil. However, as the amount of oil doubled on the return journey, the ship could carry at most 15 tons of oil. Therefore, it carried its full capacity of ethanol and mercury. On the return journey, the total load can be calculated as

$$10 + 2 \cdot 15 + \frac{1}{3} \cdot 60 = 60.$$

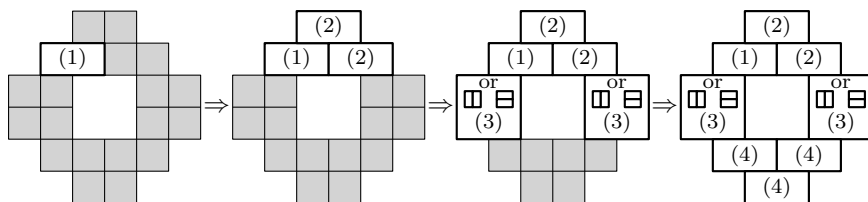
Problem 12. Determine the number of distinct ways to cover the gray shape in the diagram using non-overlapping dominoes, where each domino covers exactly two adjacent squares. A single domino (shown as a white rectangle for reference) can be rotated to fit.



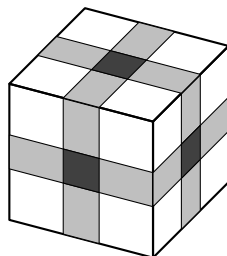
Note: Arrangements that differ by rotation or reflection of the entire shape are considered distinct, and no domino may extend beyond the boundaries of the shape.

Result. 8

Solution. Let us start covering the shape from one of the inner corners as in the diagram (1); there are two ways to place a domino, but those clearly lead to completely symmetric situations. Therefore, we can select one of them and multiply the result by 2. Once this domino is fixed, a placement of two more tiles is determined (2). The two ‘squares’ on the left and right can each be covered in two ways (3), and the remainder of the shape can then be covered uniquely (4). Hence, there are $2 \cdot 2 = 4$ ways to proceed with this placement in (1). Accounting for the symmetric option, we obtain $2 \cdot 4 = 8$ ways in total.



Problem 13. A cube-shaped package is wrapped with tape such that the width of the tape is smaller than the package edge. The dark gray regions on the surface (including those not visible in the picture) have a total area of 216 cm^2 . The area of the light gray regions on the surface is half the size of the area not covered by the tape. Determine the edge length of the package in centimeters.



Result. 30

Solution. The area of each dark gray square is $216/6 = 36$, so its side length is $\sqrt{36} = 6$. On each face, the uncovered part is twice the size of the light gray part, meaning that each light gray rectangle is half the size of one white square. Therefore we can fit 5 light gray rectangles with width 6 along one edge length of the cube, leading to the side length $5 \cdot 6 = 30$.

Problem 14. A shop sells pens, notebooks, and rulers. The price of a notebook is equal to the combined price of a pen and a ruler. If the price of a ruler were increased by 50%, it would equal the combined price of a pen and a notebook. By what percentage should the price of a pen be increased so that it equals the combined price of a notebook and a ruler?

Result. 800%

Solution. Let n be the price of the notebook, r the price of the ruler, and p the price of the pen. From the given conditions, we have the equations $n = r + p$ and $\frac{3}{2}r = p + n = 2p + r$. From the second equation, we deduce that $r = 4p$, and substituting this into the first equation gives $n = 5p$. Therefore, the price of the pen should increase ninefold, i.e., by 800%.

Problem 15. Let $\gcd(a, b)$ and $\text{lcm}(a, b)$ denote the greatest common divisor and the least common multiple of integers a and b , respectively. Evaluate the following expression:

$$\text{lcm}(2025, \text{lcm}(2024, \gcd(2023, \gcd(2022, \dots \text{lcm}(4, \gcd(3, \gcd(2, 1))) \dots)))).$$

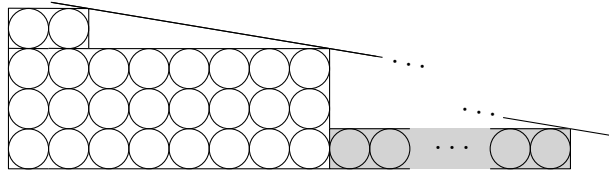
The operations \gcd and lcm alternate every two steps, with a total of 1012 \gcd computations and 1012 lcm computations in the entire expression. For example, if there were only two occurrences of each operation, the expression would be $\text{lcm}(5, \text{lcm}(4, \gcd(3, \gcd(2, 1))))$.

Result. 4 098 600

Solution. Note that $\gcd(x, x - 1) = 1$ for any positive integer x , hence $\gcd(x, \gcd(x - 1, a)) = \gcd(x, x - 1, a) = 1$ for any positive integers a and x . Therefore, the expression is equal to

$$\text{lcm}(2025, \text{lcm}(2024, 1)) = 2025 \cdot 2024 = 4\,098\,600.$$

Problem 16. In the picture, there are three rectangles with regularly inscribed congruent circles and a line passing through the upper right corners of the rectangles. The middle part of the picture is hidden. How many circles are there in the gray rectangle?



Result. 12

Solution. The right-angled triangles formed by the regions between the oblique line and the rectangles are similar, with a similarity ratio of 2. Consequently, the width of the gray rectangle is $2 \cdot 6 = 12$, measured in circle diameters.

Problem 17. Tyler ran an 18 km circuit. He started at a steady pace but, feeling fatigued at some point, he slowed his pace by 25% for the remainder of the run. After completing the run, Tyler checked his smart watch and discovered that he spent twice as much time running at the slower pace as he did at the faster pace. What distance (in kilometers) did Tyler cover before slowing down?

Result. $7.2 = \frac{36}{5}$

Solution. Let v be Tyler's original pace (in km/h) and t the time he ran at the fast pace (in hours); then his slow pace is $\frac{3}{4}v$ and the time spent at this pace is $2t$. The total distance is the sum of the two partial distances, hence

$$18 = v \cdot t + \frac{3}{4}v \cdot 2t = \frac{5}{2}vt,$$

therefore

$$vt = \frac{18}{\frac{5}{2}} = 7.2,$$

which is the distance covered at the fast pace.

Problem 18. Kathy, Laura, Megan, Natalie, and Olivia are arranging themselves into a single line for a group photo in front of a huge Náboj monument. However, there are strict conditions for who can stand where:

- Natalie must stand to the right of Kathy, Megan, and Olivia,
- Megan must stand to the left of Kathy, Natalie, and Olivia.

In how many ways can the five ladies arrange themselves for this fabulous picture?

Result. 10

Solution. Observe that Laura is totally absent in the constraints, hence she can be placed anywhere. On the other hand, there are only two ways to arrange the other four ladies, so in total, there are 10 options.

Problem 19. A castle is constructed with five towers connected by straight walls, with the wall lengths measuring 50 cubits, 70 cubits, 90 cubits, 110 cubits, and 130 cubits. The walls can be arranged in any order. What is the length (in cubits) of the longest possible straight shot an archer could make inside the castle, considering the best arrangement of the walls for this purpose?

Note: The thickness of the castle walls and the dimensions of the towers are considered negligible, and the shot distance is measured as a horizontal straight-line distance.

Result. 220

Solution. For any arrangement of castle walls, it is clear that the longest line segment inside the castle (denote its length by S) must connect two towers. Choosing these two towers divides the walls into two groups, with the sum of each group's lengths being at least S . Hence

$$S \leq \frac{50 + 70 + 90 + 110 + 130}{2} = 225$$

and since S has to be a multiple of 10, we get $S \leq 220$. This value can be achieved by splitting the walls as $130 + 90 = 220 < 110 + 50 + 70$.

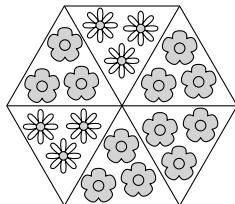
Problem 20. Pauline has 8 cards, each labelled with a unique digit from 1 to 8. She arranges all the cards to create two four-digit numbers. What is the smallest possible positive difference between these two numbers?

Result. 247

Solution. The difference is smallest when the numbers are as close to each other as possible. For this purpose, the thousands digit may only differ by 1. The hundreds digit must be as small as possible for the larger number and as large as possible for the smaller number. Once the hundreds digit is fixed, the same applies to the tens digit and then finally to the units digit. This leads to the numbers 5123 and 4876, whose difference is 247.

Problem 21. Grandma Joan has decided to plant a hexagonal arrangement of six flower beds using two types of flowers: violets and daisies. Each of the six beds positioned in a regular hexagon can be planted with either violets or daisies. One such arrangement is shown in the picture. In how many ways can she arrange the plantings so that there is at least one pair of adjacent beds with the same type of flower?

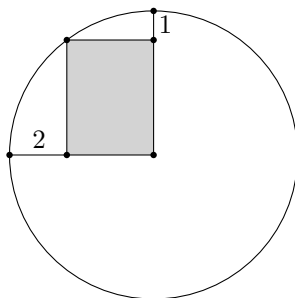
Note: Arrangements that differ by any symmetry (rotation or reflection) are still considered different. Each of the six flower-bed positions is treated as distinct.



Result. 62

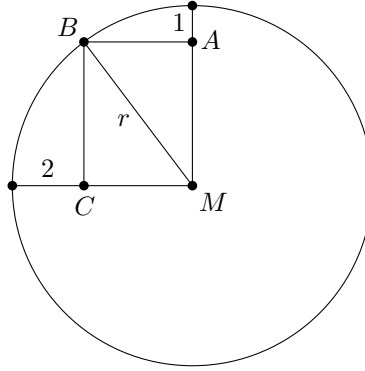
Solution. If we drop the condition that two neighbouring beds have to be planted with the same type of flowers, then the answer is $2^6 = 64$. From this result, we subtract the options where the condition is violated, which is the case when the two types of flowers alternate. Those being only two options, this leads to the total of $64 - 2 = 62$ valid arrangements.

Problem 22. From a circular sheet of paper, Erich cut out the rectangular piece so that one corner of the rectangle is located at the center of the circle, the opposite corner lies on the circumference of the circle, and the remaining two corners lie on two distinct radial lines extending from the center, positioned at distances of 1 dm and 2 dm from the circumference along these lines. What is the area of the circular sheet that remains after the cut (in dm^2)?



Result. $25\pi - 12$

Solution. Let M be the center of the circular sheet, and let A, B, C be the remaining vertices of the rectangle. Denote the radius of the circle as r .



We have $MA = r - 1$, $MB = r$, and $MC = r - 2$, with $AB = MC$. Applying the Pythagorean theorem in the right-angled triangle ABM gives the equation

$$r^2 = (r - 1)^2 + (r - 2)^2,$$

which can be simplified to

$$0 = r^2 - 6r + 5 = (r - 1)(r - 5).$$

Since $r = 1$ leads to an invalid configuration, the only valid solution is $r = 5$. The remaining area of the sheet is $r^2\pi - 3 \cdot 4 = 25\pi - 12$ (in dm^2).

Problem 23. Grandmaster Náboicus, the unparalleled virtuoso of essence blending, is about to craft the legendary *Algemy*—a flawless fusion of Algebra and Alchemy, mixed in a perfect 1 : 1 ratio. To achieve this, he begins with the following ingredients:

- Algery: Composed of 80% Algebra and 20% Alchemy, with a total mass of 10 mg,
- Alchema: Composed of 30% Algebra and 70% Alchemy, with a total mass of 14 mg.

With these resources at his disposal, how much Algemy can Náboicus produce at most (in mg)?

Note: Náboicus cannot isolate the components of a mixture at any point during the process, he can only further mix the available substances.

Result. $23\frac{1}{3} = \frac{70}{3}$

Solution. If we mix x units of the former substance and y of the latter, the resulting mixture contains $\frac{4}{5}x + \frac{3}{10}y$ Algebra and $\frac{1}{5}x + \frac{7}{10}y$ Alchemy. To obtain 1 : 1 ratio,

$$\frac{4}{5}x + \frac{3}{10}y = \frac{1}{5}x + \frac{7}{10}y$$

must hold, which can be rearranged to $y = \frac{3}{2}x$. In other words, for each mg Algery, 1.5 mg of Alchema have to be used in the mixture. Hence the maximal amount of Algemy will be produced when Náboicus uses all 14 mg of Alchema and $\frac{2}{3} \cdot 14$ mg of Algery, producing $\frac{5}{3} \cdot 14 = \frac{70}{3}$ mg of the sought mixture in total.

Problem 24. A number $K = n^2$ is a 4-digit perfect square with all its digits less than 7. If each digit of K is increased by 3, another perfect square is obtained. Find n .

Result. 34

Solution. Let $m^2 = M = K + 3333$. Since M and K are both 4-digit perfect squares, we get $32 \leq n < m \leq 99$ and therefore

$$32 + 33 \leq m + n \leq 98 + 99$$

or

$$65 \leq m + n \leq 197.$$

Now

$$(m + n)(m - n) = m^2 - n^2 = M - K = 3333 = 3 \cdot 11 \cdot 101.$$

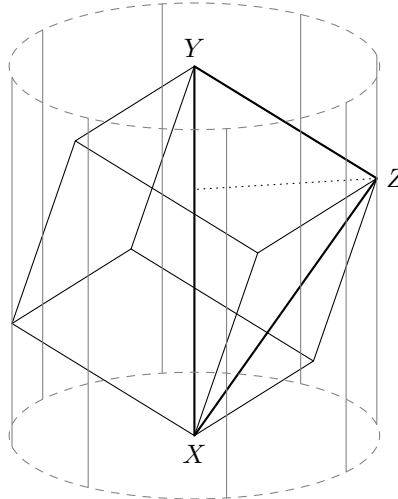
Given the constraints above, the only possible factors are $m + n = 101$ and $m - n = 33$, yielding the solution $m = 67$ and $n = 34$. It remains to check that $K = 34^2 = 1156$ indeed has all its digits less than 7 in this case.

Problem 25. Let X, Y be two opposite vertices of a cube of sidelength 1 and let C be a right cylinder whose surface contains all the vertices of the cube so that X and Y are centers of the circular bases of C . What is the volume of C ?

Result. $\frac{2\pi\sqrt{3}}{3} = \frac{2\pi}{\sqrt{3}}$

Solution. As X, Y are centers of the bases of the cylinder, the height of the cylinder equals their distance. Since X and Y are opposite vertices of the cube, they lie on a space diagonal of length $\sqrt{3}$. To determine the radius of the cylinder, pick any other vertex Z of the cube and compute its distance from the diagonal XY . The triangle XZY is right-angled at Z (XZ is a face diagonal and YZ is an edge of the cube), and the sought radius is the altitude from Z onto XY . By similarity (or area comparisons), this altitude is $\sqrt{2/3}$. Hence, the volume of the cylinder is

$$\pi \left(\sqrt{\frac{2}{3}} \right)^2 \sqrt{3} = \frac{2\pi\sqrt{3}}{3}.$$



Problem 26. In a village of 60 people, each person belongs to one of three types: truth-tellers, who always tell the truth; liars, who always lie; and normal people, who are free to answer however they prefer. Everyone in the village knows everyone else's type. An outsider asked all the villagers two questions:

1. "Are there at least 31 truth-tellers?" – and received exactly 43 positive replies.
2. "Are there at least 31 liars?" – and received exactly 39 positive replies.

What is the minimum number of normal people in the village?

Result. 13

Solution. The second assertion cannot be true: If there were at least 31 liars, they would all respond to the question negatively, making it impossible to have 39 positive replies. Therefore, there are at most 30 liars. Since truth-tellers always tell the truth, they must have answered this question negatively, meaning there can be at most $60 - 39 = 21$ truth-tellers.

This also implies that the first assertion is false. The 43 positive responses must have come from all the liars plus some normal people. Since there are at most 30 liars, at least $43 - 30 = 13$ normal people must exist.

This configuration is feasible, as the distribution 17 truth-tellers, 30 liars, and 13 normal people satisfies both conditions. Thus, the minimum number of normal people in the village is 13.

Problem 27. Four teams, A, B, C , and D , participated in a round-robin tournament where each pair of teams played exactly one match. The winner of each match was awarded either 1 or 2 points, depending on the margin of their victory, while the losing team did not get any points. There were no draws. After all the matches, a table like the example below was created, showing the results of all the matches. If we know that one team ended with 4 points, while the remaining three teams each had 1 point, how many such tables could fit this final scoring distribution?

| | A | B | C | D | total |
|-----|-----|-----|-----|-----|-------|
| A | | 1 | 0 | 2 | 3 |
| B | 0 | | 0 | 0 | 0 |
| C | 2 | 1 | | 2 | 5 |
| D | 0 | 1 | 0 | | 1 |

Note: The arrangement of the teams (A, B, C , and D) in the table is fixed, meaning that the labels for rows and columns do not change.

Result. 24

Solution. First, observe that the total number of points awarded is 7, which means that out of the 6 matches, exactly one resulted in the winning team obtaining two points, while the rest resulted in the victors earning only one point each. This implies that the best team defeated all the other teams, with exactly one of those victories being by the larger margin. The matches among the three non-top teams must have resulted in a “cycle”, since each of these teams was awarded only one point, and there are exactly two ways to orient this cycle. In total, there are $4 \cdot 3 \cdot 2 = 24$ ways the tournament could have unfolded.

Problem 28. Julia wrote the number 2025 as a sum of M terms, where each term is a power of 10 (i.e., 10^n , where n is a non-negative integer). The terms in the sum may repeat. How many different values can M take?

Result. 225

Solution. Clearly, the smallest possible value of M is 9, because

$$2025 = 2 \cdot 10^3 + 2 \cdot 10^1 + 5 \cdot 10^0.$$

If $k \geq 1$, then each substitution $10^k = 10 \cdot 10^{k-1}$ increases the number of summands by 9. This implies that M has to be a multiple of 9; moreover, the possible values of M form a consecutive set of multiples of 9. The largest possible value of M is 2025, because

$$2025 = 2025 \cdot 10^0.$$

Thus, the number of possible values for M is

$$\frac{2025 - 9}{9} + 1 = 225.$$

Problem 29. Max and Paul are standing back-to-back on a train station platform. A freight train traveling at a constant speed passes by them. The moment the front of the train aligns with their position, both Max and Paul start walking in opposite directions at the same constant speed. The rear of the train reaches Max when he is 45 meters from his starting point, and shortly afterwards, it reaches Paul when he is 60 meters from his starting point. How long is the train in meters?

Result. 360

Solution. Let t_1 be the time elapsed from the moment the front of the train passes Max and Paul until the rear of the train passes Max. Similarly, let t_2 be the time elapsed from the moment the rear of the train passes Max until it passes Paul. Finally, let ℓ be the length of the train in meters. Since Max and Paul walk at the same speed, and during t_1 , Max managed to cover 45 meters while Paul covered $60 - 45 = 15$ meters during t_2 , the ratio of the time intervals is

$$t_1 : t_2 = 45 : 15 = 3 : 1.$$

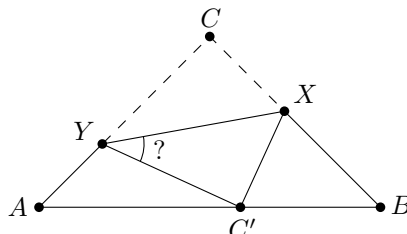
Now, consider the motion of the train. During t_1 , the train advances by $\ell - 45$ meters, because initially, its front was at the meeting point, while at the end of this interval, the rear was still 45 meters away from that point. During t_2 , the rear of the train covers the 105 meters between Max and Paul. Thus, the speed of the train is

$$v = \frac{105}{t_2}.$$

The full length of the train is then given by

$$\ell = vt_1 + 45 = \frac{t_1}{t_2} \cdot 105 + 45 = 3 \cdot 105 + 45 = 360.$$

Problem 30. An isosceles right triangle ABC with a right angle at C is folded along a segment XY such that vertex C is mapped to a point C' on side AB . Additionally, it is given that $BC' = BX$. Find the measure of angle $C'YX$ in degrees.



Result. $33.75^\circ = \frac{135}{4}^\circ$

Solution. Since triangle $XC'B$ is isosceles, we have

$$\angle C'XB = \frac{1}{2}(180^\circ - 45^\circ) = 67.5^\circ.$$

Furthermore, $\angle CXY = \angle YXC'$ because of the folding, hence

$$\angle YXC' = \frac{1}{2}(180^\circ - 67.5^\circ) = 56.25^\circ.$$

Finally, $\angle XC'Y = \angle YCX = 90^\circ$, thus

$$\angle C'YX = 180^\circ - \angle XC'Y - \angle YXC' = 33.75^\circ.$$

Problem 31. Consider the sequence of all strictly increasing 4-tuples with entries in the set $\{0, 1, 2, \dots, 15\}$, arranged in lexicographic order:

$$(0, 1, 2, 3), (0, 1, 2, 4), (0, 1, 2, 5), \dots, (12, 13, 14, 15).$$

That is, (a_1, a_2, a_3, a_4) appears earlier than (b_1, b_2, b_3, b_4) in this sequence if and only if

$$a_1 < b_1 \quad \text{or} \quad a_1 = b_1, a_2 < b_2 \quad \text{or} \quad a_1 = b_1, a_2 = b_2, a_3 < b_3 \quad \text{or} \quad a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 < b_4.$$

Find the position of the tuple $(2, 4, 7, 14)$ in the sequence.

Result. 911

Solution. Observe that, for $k \leq n$, the number of all increasing k -tuples with entries in the set $\{1, 2, \dots, n\}$ equals $\binom{n}{k}$, since strictly increasing tuples directly correspond to subsets of size k . More generally, the number of all increasing k -tuples with entries in the set $\{m, m+1, \dots, n\}$ equals $\binom{n-m+1}{k}$. To determine the position of the tuple, we count the number of preceding increasing 4-tuples, grouping them based on their known entries:

- $(0, *, *, *)$: there are $\binom{15}{3} = 455$ such tuples;
- $(1, *, *, *)$: there are $\binom{14}{3} = 364$ such tuples;
- $(2, 3, *, *)$: there are $\binom{12}{2} = 66$ such tuples;
- $(2, 4, a, *)$ with $a \in \{5, 6\}$: there are $\binom{10}{1} + \binom{9}{1} = 19$ such tuples;
- $(2, 4, 7, b)$ with $b \in \{8, 9, 10, 11, 12, 13\}$: there are 6 such tuples.

Thus, the tuple $(2, 4, 7, 14)$ appears at position $455 + 364 + 66 + 19 + 6 + 1 = 911$.

Problem 32. Two farmers, Adam and Bettina, sold apples at the market. Together, they brought a total of 100 apples. Adam sold his apples at a price of a coins per apple, while Bettina sold hers at b coins per apple. After selling all their apples, they each earned the same total amount. Adam then remarked that if he had sold his apples at Bettina's price of b coins per apple, he would have earned 45 coins. Bettina added that if she had sold her apples at Adam's price of a coins per apple, she would have earned 20 coins. What is the number of apples sold by Adam?

Result. 60

Solution. Let A and B denote the number of apples brought to the market by Adam and Bettina respectively. We know that

$$\begin{aligned} A + B &= 100, \\ A \cdot a &= B \cdot b, \\ A \cdot b &= 45, \\ B \cdot a &= 20. \end{aligned}$$

Substituting $b = \frac{45}{A}$ and $a = \frac{20}{B}$ into the second equation, we get

$$\begin{aligned} A \cdot \frac{20}{B} &= B \cdot \frac{45}{A}, \\ \frac{A^2}{B^2} &= \frac{45}{20} = \frac{9}{4}. \end{aligned}$$

Therefore $A = \frac{3B}{2}$, and substituting into the first equation we get $\frac{3B}{2} + B = \frac{5B}{2} = 100$, therefore $B = 40$ and $A = 100 - 40 = 60$.

Problem 33. Sue dreamed of a fascinating number. It is the largest three-digit number with a unique property: it equals the sum of its hundreds digit, the square of its tens digit, and the cube of its units digit. Can you determine the number Sue dreamed of?

Result. 598

Solution. Let a , b , and c be the hundreds, tens, and units digits of the three-digit number N . If $c = 9$, then $N > 9^3 = 729$, so a must be 7 or 8. However, in both cases, there is no choice of b making the last digit of $729 + a + b^2$ equal to 9.

Next, consider $c = 8$. Since $8^3 = 512$, a must be 5 or 6, but

$$N \leq 512 + 9^2 + 6 = 599 < 600,$$

so a can only be 5. We then need b satisfying

$$512 + b^2 + 5 = 8 + 10b + 500$$

or

$$b^2 - 10b + 9 = 0,$$

whose solutions are $b = 1$ and $b = 9$. Both yield valid numbers:

$$518 = 5 + 1^2 + 8^3 \quad \text{and} \quad 598 = 5 + 9^2 + 8^3.$$

(Equivalently, one can note the units digit of b^2 must be 1.)

If $c \leq 7$, then

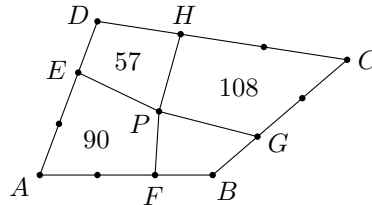
$$N \leq 7^3 + 9^2 + 9 = 433 < 598,$$

so the largest valid value of N is 598.

Problem 34. Each side of a quadrilateral $ABCD$ is divided into three equal parts by two points. Specifically:

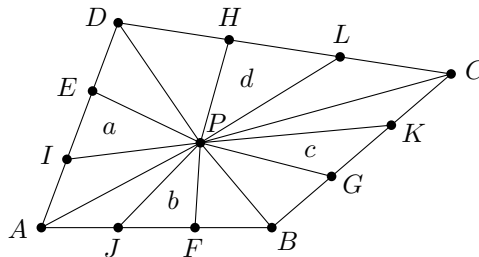
- Point E lies on segment AD , such that $AE : ED = 2 : 1$.
- Point H lies on segment CD , such that $CH : HD = 2 : 1$.
- Point F lies on segment AB , such that $AF : FB = 2 : 1$.
- Point G lies on segment CB , such that $CG : GB = 2 : 1$.

A point P lies inside the quadrilateral, splitting it into four smaller quadrilaterals. The areas of three of these quadrilaterals are given in the diagram. Determine the area of the fourth quadrilateral, $PFBG$.



Result. 42

Solution. Connecting P to all points that divide the sides of quadrilateral $ABCD$ into thirds creates twelve triangles.



Each set of three triangles along the same side has equal area because they share the same height from point P and have bases of equal length. Let a be the area of triangle PEI , b be the area of triangle PJI , c be the area of triangle PGK and d be the area of triangle PLH . From the given information, we establish the equations

$$90 = 2a + 2b, \quad 57 = a + d, \quad 108 = 2c + 2d.$$

The area of quadrilateral $PFBG$ is given by $b + c$. This can be computed as

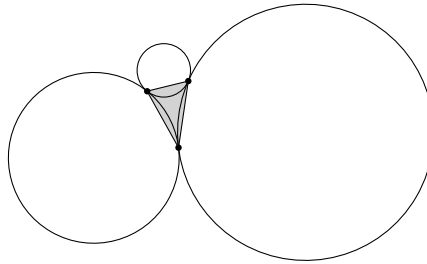
$$b + c = \frac{1}{2}(2a + 2b + 2c + 2d) - (a + d) = \frac{1}{2}(90 + 108) - 57 = 42.$$

Problem 35. A ring of twelve squares is formed by removing the four central squares from a 4×4 board. In how many ways can four squares be chosen in the ring such that at least one square is selected from each side of the ring? Note: Each corner square belongs to two sides. Choices differing only by symmetry (rotations or reflections of the ring) are considered distinct.

Result. 237

Solution. Overall there are $\binom{12}{4} = 495$ ways of choosing 4 squares from a 12-square ring. For each of the four sides, there are eight squares not belonging to that side, giving $\binom{8}{4} = 70$ ways of choosing four squares so that this side is left out. Therefore, there would be $495 - 4 \cdot 70 = 215$ choices so that no side is left out. However, some choices were subtracted twice – namely those where two sides are left out at once. This is possible to do by choosing 4 out of 5 squares grouped around a corner (5 ways for a corner, 20 for all 4 corners) or 4 out of 4 opposite middle squares (two ways overall). This gives us $215 + 22 = 237$ ways. Because omitting three or four sides is impossible in this situation, this is the final answer.

Problem 36. Three circles with radii 1, 2, and 3, respectively, are externally tangent to each other, as shown in the figure. Determine the area of the triangle formed by the three points of tangency.



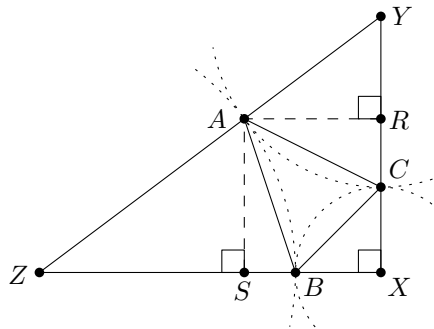
Result. $\frac{6}{5}$

Solution. Denote the centres of the circles X, Y, Z , respectively, and let further be A, B, C the points of tangency as in the diagram below. Triangle XYZ has the side lengths $1 + 2 = 3$, $1 + 3 = 4$, and $2 + 3 = 5$, which is a triple of numbers satisfying the Pythagorean theorem, so it is a right triangle with right angle at X . To compute the sought area of triangle ABC , we will subtract the areas of isosceles triangles XCB , YAC , and ZBA from the area of triangle XYZ , which is $\frac{1}{2}(3 \cdot 4) = 6$.

- Triangle XCB is right-angled, therefore its area is $\frac{1 \cdot 1}{2} = \frac{1}{2}$.
- To compute the area of triangle YAC , we first find the length of its height AR . Since triangles RYA and XYZ are similar with similarity ratio $YA/YZ = \frac{2}{5}$, it follows that $AR = \frac{2}{5}ZX = \frac{8}{5}$. Therefore, the area of triangle YAC is $\frac{1}{2}(\frac{8}{5} \cdot 2) = \frac{8}{5}$.
- Similarly, to get the area of triangle ZBA , we utilize the similarity $\triangle SAZ \sim \triangle XYZ$ with ratio $\frac{3}{5}$ to get $AS = \frac{9}{5}$, so the area of triangle ZBA is $\frac{1}{2}(\frac{9}{5} \cdot 3) = \frac{27}{10}$.

Finally, we can calculate the desired area as

$$6 - \frac{1}{2} - \frac{8}{5} - \frac{27}{10} = \frac{6}{5}.$$



Problem 37. Agnes drew a regular n -gon with $n > 3$ and counted its diagonals. She observed that the total number of diagonals was a multiple of 2025. What is the smallest possible value of n that satisfies this condition?

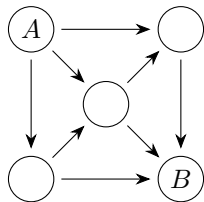
Note: The sides of the n -gon are not counted as diagonals.

Result. 300

Solution. It is easy to see that the number of diagonals of an n -gon is $\frac{1}{2}n(n-3)$. As 2025 is odd, we can equivalently investigate when the product $P = n(n-3)$ is divisible by 2025. As $2025 = 3^4 \cdot 5^2$, thanks to coprimality, we have to ensure that P is divisible by $3^4 = 81$ and $5^2 = 25$. Note that only one of the factors $n, n-3$ can be divisible by 5, so one of them must be divisible by 25. On the other hand, n is divisible by 3 if and only if $n-3$ is divisible by 3, so both of them contribute to the total power of 3 dividing P ; however, only one of $n, n-3$ can be divisible by 3^k for $k \geq 2$. This means that we need one of the factors to be divisible by $3^3 = 27$.

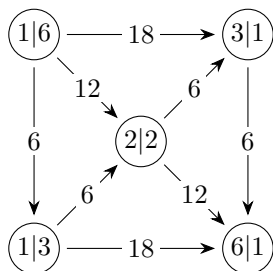
If one of the factors was divisible by both 25 and 27, it would be equal to at least $25 \cdot 27 = 675$. Let us check whether a smaller value can be found by choosing one of the factors (say m) to be divisible by 27 and the other one ($m \pm 3$) to be divisible by 25 (this way either $n = m$ or $n = m + 3$). By the first condition, we have $m = 27k$ for some positive integer k and we are interested in the smallest value of k such that $27k \pm 3$ is divisible by 25 (for one of the sign choices). As $25k$ is always divisible by 25, we can equivalently check $2k \pm 3$, which is first divisible by 25 for $k = 11$ and the positive sign. Therefore $m = 27 \cdot 11 = 297$ and $n = m + 3 = 300$.

Problem 38. David embarks on a journey along the paths in the diagram below. He starts at node A and ends at node B . He must follow the direction of the arrows on the diagram, except for one rebellious move where he deliberately travels against the direction of an arrow. This rebellious move must occur exactly once during his journey, even if it means temporarily leaving the final destination. David is allowed to use any arrow more than once as he navigates the diagram. In how many distinct ways can David complete his journey under these conditions?



Result. 84

Solution. For each node we compute (1) the number of (oriented) paths from A ending at it, (2) the number of paths to B starting from it. In the diagram below, for example, $3|1$ in the top right node indicates that there are exactly three paths from the starting node that end at this node, and only one path from this node to the final node. For each arrow, the number of paths traversing it in the opposite direction is given by the product of the first number at the endpoint and the second number at the starting point, so we compute this product for every arrow and sum up. The result is 84.



Problem 39. In the following computation, different letters stand for different non-zero digits.

$$\begin{array}{r}
 N \quad N \quad N \quad N \quad N \\
 + \quad A \quad A \quad A \quad A \\
 \quad \quad + \quad B \quad B \quad B \\
 \quad \quad \quad + \quad O \quad O \\
 \quad \quad \quad \quad + \quad J \\
 - \quad 2 \quad 0 \quad 2 \quad 5 \\
 \hline
 = \quad N \quad A \quad B \quad O \quad J
 \end{array}$$

Determine the largest possible value of the five-digit number $NABOJ$.

Result. 18249

Solution. We can rewrite the computation equivalently as

$$\begin{array}{r}
 N \quad N \quad N \quad N \quad N \\
 + \quad A \quad A \quad A \quad A \\
 \quad \quad + \quad B \quad B \quad B \\
 \quad \quad \quad + \quad O \quad O \\
 \quad \quad \quad \quad + \quad J \\
 - \quad N \quad A \quad B \quad O \quad J \\
 \hline
 = \quad 2 \quad 0 \quad 2 \quad 5
 \end{array}$$

which further simplifies to

$$\begin{array}{r}
 N \quad N \quad N \quad N \\
 + \quad A \quad A \quad A \\
 \quad \quad + \quad B \quad B \\
 \quad \quad \quad + \quad O \\
 \hline
 = \quad 2 \quad 0 \quad 2 \quad 5
 \end{array}$$

It follows that $N = 1$. At this point, we are left with $\overline{AAA} + \overline{BB} + O = 2025 - 1111 = 914$, implying $A = 8$. Further reduction $914 - 888 = 26$ yields $B = 2$ and finally, $O = 4$. The value of J can be arbitrary, but different from the digits already used; hence, the largest possible value of $NABOJ$ is 18249.

Problem 40. Five hundred Náboj organizers were voting on competition problems. For each problem, every organizer present voted either in favour of it or against it. However, after just the first problem, some organizers who voted in favour of the problem found the process so tiresome that they decided to leave the room. At the same time, none of those voting against the first problem left. In the vote on the second problem, the same number of organizers voted in favour of it as in the first vote, but the number of votes against the problem was only one-third of the votes against the first problem. Additionally, it is known that exactly 120 organizers voted in favour of both problems and 70 voted against both problems. How many organizers left the room after the first vote?

Result. 150

Solution. Denote by YN the number of organizers voting for in the first voting and against in the second; similarly define YY , NN , and NY . Finally, let YX be the number of leaving organizers. Then we have the following system of equations:

$$\begin{aligned}
 YY + YN + NY + NN + YX &= 500 \\
 YY + YN + YX &= YY + NY \\
 NY + NN &= 3(YN + NN)
 \end{aligned}$$

Substituting $YY = 120$ and $NN = 70$ and rearranging the terms, we get

$$\begin{array}{rclcl}
 YN & + & NY & + & YX & = & 310 \\
 YN & - & NY & + & YX & = & 0 \\
 -3YN & + & NY & & & = & 140
 \end{array}$$

Multiplying the second equation by 2 and summing all equations together yields $3YX = 450$, hence $YX = 150$. The remaining two variables are equal to $YN = 5$ and $NY = 155$.

Problem 41. Determine the number of pairs (a, b) of positive integers satisfying $a \leq b$ and such that $\gcd(a, b)$, together with a and b , can be arranged into an arithmetic sequence whose sum is 2025.

Note: An arithmetic sequence is a sequence of numbers where the difference between one number and the next one is always the same.

Result. 12

Solution. Let $g = \gcd(a, b)$. Then $a = ga'$, $b = gb'$ for some positive integers a' , b' . Since $g \leq a \leq b$, the three-term arithmetic sequence is (g, a, b) or its reverse; in either case, $a - g = b - a$ or $b = 2a - g$, which becomes $b' = 2a' - 1$ after dividing by g . From the condition on the sum we obtain

$$g + a + b = g(1 + a' + 2a' - 1) = 3ga' = 2025,$$

hence $ga' = 675 = 3^3 5^2$. This number has $(3 + 1) \cdot (2 + 1) = 12$ positive divisors and it remains to check that each such divisor yields a valid value of a' , i.e., one that can be completed to a valid pair (a, b) . Indeed, by letting $b' = 2a' - 1$, $g = 675/a'$, $a = ga' = 675$, $b = gb' = 675(2a' - 1)$, we have $a \leq b$ since $ga' \leq g(2a' - 1)$ for any positive integers a' , g , and also

$$\gcd(a, b) = \gcd(ga', g(2a' - 1)) = g \cdot \gcd(a', 2a' - 1) = g$$

as a' and $2a' - 1$ are coprime for any positive integer a' .

Problem 42. Each team in Kindergarten Náboj is initially given the first 3 problems from a pool of 16 numbered problems. Every team has its own pool but all the pools contain the same 16 problems numbered in the same way. When a team solves a problem, it gets replaced by the problem from that team's problem pool with the lowest available number. After the competition, it turned out that no two teams solved exactly the same set of problems. What is the maximal number of teams that took part in this competition?

Result. 697

Solution. Note that the set of problems solved by a team is fully determined by the set of unsolved problems and vice versa; this can be also viewed as the set of the problems the team has left on the table at the end of the contest, which can be any set of size at most 3. Therefore there are at most

$$\binom{16}{0} + \binom{16}{1} + \binom{16}{2} + \binom{16}{3} = 697$$

teams competing.

Problem 43. Let a, b, c, d be real numbers such that

$$2a + 2b - ab = 2025,$$

$$2b + 2c - bc = 47,$$

$$2c + 2d - cd = 5.$$

Find the value of $2a + 2d - ad$.

Result. 51

Solution. Using the formula $(x-2)(y-2) = xy - 2x - 2y + 4$, the given conditions can be rearranged to the following equations:

$$(a-2)(b-2) = -2021,$$

$$(b-2)(c-2) = -43,$$

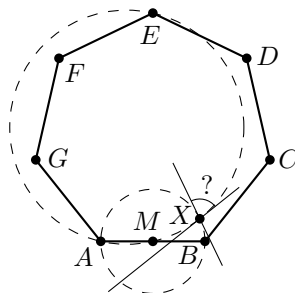
$$(c-2)(d-2) = -1$$

and our goal is to find $(a-2)(d-2)$. Since $b \neq 2$ and $c \neq 2$ because of the second equation, the desired expression can be obtained as

$$(a-2)(d-2) = \frac{(a-2)(b-2)(c-2)(d-2)}{(b-2)(c-2)} = \frac{(-2021) \cdot (-1)}{-43} = -47.$$

Therefore, $2a + 2d - ad = -(-47) + 4 = 51$.

Problem 44. Let M be the midpoint of the side AB of a regular heptagon $ABCDEFGH$. Circle centered at M and passing through A intersects the circumcircle of triangle AME at a point X lying inside the heptagon. What is the size (in degrees) of the acute angle between the tangents to the two circles at X ?



Result. $540^\circ/7$

Solution. Since $ABCDEFGH$ is a regular heptagon, the angle AME is a right angle and AE is the diameter of the larger circle. Instead of measuring the angle between the tangents at X , we can equivalently consider the angle between the tangents at A , which is the second intersection point of the two circles. This angle is in turn equal to the angle BAE between the corresponding diameters, since those are perpendicular to the tangents. Its size, $\frac{3}{7} \cdot 180^\circ$, can be easily determined from the symmetry of the regular heptagon or by recognizing that it is the inscribed angle corresponding to the central angle of $\frac{3}{7} \cdot 360^\circ$ on the circumcircle of the heptagon.

Problem 45. Compute the sum of the values $(\pm 1 \pm 2 \pm 4 \pm \dots \pm 2^{99})^2$ over all possible choices of the 100 signs.

Result. $\frac{2^{100} \cdot (4^{100} - 1)}{3}$

Solution. We begin with a more general observation: for any positive integer n and real numbers x_1, x_2, \dots, x_n , if we sum the squared expressions $(\pm x_1 \pm x_2 \pm \dots \pm x_n)^2$ over all possible choices of signs, the result is always

$$2^n(x_1^2 + x_2^2 + \dots + x_n^2).$$

To see why this holds, consider expanding $(\pm x_1 \pm x_2 \pm \dots \pm x_n)^2$. Each expansion consists of squared terms x_i^2 , as well as mixed terms of the form $\pm 2x_i x_j$ for $i \neq j$. Each term x_i^2 appears in every possible expansion, regardless of the chosen signs. Since there are 2^n different sign combinations, these squared terms contribute a total factor of 2^n . On the other hand, the mixed terms $\pm 2x_i x_j$ appear with a positive sign in exactly half of the cases and with a negative sign in the other half, depending on whether x_i and x_j have the same sign or not. Since these contributions perfectly cancel out across all sign choices, they do not affect the final sum. Thus, the total sum simplifies to 2^n times the sum of the squared terms, proving the formula.

In our case, we have $x_k = 2^{k-1}$, and the desired sum equals

$$2^{100} \cdot (1 + 4^1 + \dots + 4^{99}).$$

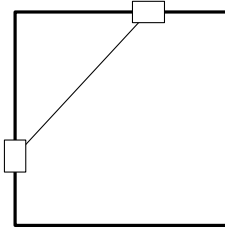
Using the formula for the sum of a geometric series,

$$1 + 4^1 + \dots + 4^{99} = \frac{4^{100} - 1}{4 - 1} = \frac{4^{100} - 1}{3},$$

we obtain the final result

$$\frac{2^{100} \cdot (4^{100} - 1)}{3}.$$

Problem 46. Two cars, connected by a rubber band, are traveling along a square-shaped road, as shown in the picture. Initially, both cars start together at one corner of the square. Each car then travels indefinitely at a constant integer speed. The rubber band is extremely elastic but will break if stretched across the exact diagonal of the square. The slower car moves at a speed of 24 km/h, while the faster car travels at a speed of n km/h, both in the same direction. Determine the smallest integer value of n greater than 24 such that the rubber band will never break.



Result. 56

Solution. We define a *segment* as one side of the square road. Let $m = 24$ and $n > m$ be the speeds of the slower and faster cars, respectively. Observe that whether the rubber band ever breaks depends only on the ratio of the speeds, not on their absolute values. Thus, let m', n' be coprime positive integers with $m' : n' = m : n$. We claim that the band never breaks precisely when $n' - m'$ is a multiple of 4.

First, let us analyze the case when $n' - m'$ is not a multiple of 4. After the slower car has traveled m' segments, it is at a corner. At that same time, the faster car has traveled n' segments (due to the ratio of speeds), so it is at a corner, too. Because $n' - m'$ is not divisible by 4, these two corners cannot coincide. If they turn out to be opposite corners, the band breaks immediately. If they are neighboring corners, then after traveling another m' and n' segments, the cars end up at opposite corners, causing the band to break as well.

Let us now assume that $n' - m'$ is a multiple of 4. First, note that the cars cannot both be at a corner any earlier than when the slower car has traveled m' segments. Otherwise, say after some $s < m'$ segments of the slower car, the faster car would have traveled $\frac{s}{m'} \cdot n'$ segments, which cannot be an integer because m' and n' are coprime. Hence, the first time both cars reach corners simultaneously is when the slower car has traveled m' segments and the faster car n' . By assumption, $n' - m'$ is a multiple of 4, so they must land at the same corner. Once they coincide at a corner, the entire situation effectively resets (possibly starting from a different corner), so the band never breaks.

It remains to find the smallest $n > 24$ that satisfies the given condition. We must have $n' - m'$ divisible by 4. In that case, both m' and n' must be odd. Since m' is a divisor of $m = 24$, the only possible values are 1 and 3. If $m' = 1$, the smallest n' that is coprime to m' and such that $n' - 1$ is a multiple of 4 is 5. Then $n = \frac{24}{1} \cdot 5 = 120$. If $m' = 3$, the smallest n' coprime to m' with $n' - 3$ divisible by 4 is 7. Here, $n = \frac{24}{3} \cdot 7 = 56$. Since 56 is smaller than 120, we conclude that the desired smallest value of n is 56.

Problem 47. At a table, 2025 players are playing a game. At the end of each round, the losing player gives each of the other players a number of coins equal to the number they currently possess (so different players can receive different amounts of money). After 2025 rounds, each player has exactly 2^{3000} coins. Moreover, no player had debts at any point during the game. If each player lost exactly one round, determine the initial number of coins held by the player who lost the first round.

Result. $2^{975} + 2025 \cdot 2^{2999} = 2^{975} \cdot (1 + 2025 \cdot 2^{2024})$

Solution. Let us denote the players by numbers $1, 2, \dots, 2025$, and the amount of coins owned by player p after r rounds as $m_{p,r}$. Without loss of generality, assume that player 1 lost the first round, player 2 the second one, and so on. Denote $M = 2^{3000}$, the number of coins at the end of the game held by all players.

For a player p after their loss at p -th round, the amount of their coins is doubling till it reaches M at the end of the game; hence for a round $p \leq r$, the amount of coins is

$$m_{p,r} = \frac{M}{2^{2025-r}}.$$

Moreover, since the r -th player lost the r -th round, they lost the number of coins equal to the amount held by all other players, and since the total amount of coins in the game is $2025M$, it follows that

$$m_{r,r} = m_{r,r-1} - \sum_{p \neq r} m_{p,r-1} = m_{r,r-1} - (2025M - m_{r,r-1}).$$

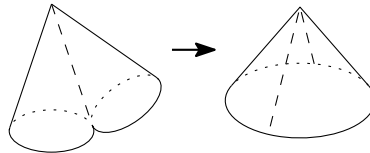
By rearranging the equation, we get

$$m_{r,r-1} = \frac{m_{r,r}}{2} + \frac{2025M}{2} = \frac{M}{2^{2026-r}} + \frac{2025M}{2}.$$

Plugging in $r = 1$, we obtain the desired result

$$\frac{M}{2^{2025}} + \frac{2025M}{2} = 2^{975} + 2025 \cdot 2^{2999} = 2^{975} \cdot (1 + 2025 \cdot 2^{2024}).$$

Problem 48. Gleb has three identical paper models of the lateral surface of a right cone (excluding the base). The base of the cone is a circular disc perpendicular to the axis connecting its center to the cone's vertex, but this disc is not part of the paper models. First, Gleb placed two of the cone surfaces vertex-to-vertex so that they shared a line segment along their lateral surfaces. He cut both along this segment and joined the two surfaces to create a larger cone surface (as shown in the picture). The volume of the full cone corresponding to this larger surface was 10. Next, Gleb joined this larger cone surface with the third original cone surface in the same way, expecting to measure the volume of the resulting cone. However, he realized the resulting volume was zero. What was the volume of the original cone?



Result. $\sqrt{10}$

Solution. The fact that the final cone has zero volume means that joining three identical lateral surfaces results in a completely flat shape – a full circle. Consequently, each individual cone surface, when flattened, corresponds to a circular sector with a central angle of 120° . Let l be the slant height of the original cone and r its base radius. From the central angle, we obtain the relation $l = 3r$. Observe that the slant height remains unchanged across all three cones, including the degenerate one.

The intermediate cone, formed by joining two lateral surfaces, has a central angle of 240° , giving it a base radius of $2r$. Applying the cone volume formula, we find

$$10 = \frac{1}{3}\pi(2r)^2\sqrt{(3r)^2 - (2r)^2} = \frac{4\sqrt{5}}{3}\pi r^3.$$

From this, we compute the volume of the original cone as

$$\frac{1}{3}\pi r^2\sqrt{(3r)^2 - r^2} = \frac{2\sqrt{2}}{3}\pi r^3 = \sqrt{10}.$$

Problem 49. For how many positive integers n less or equal to 200 has the equation

$$5 \left\lfloor \frac{x}{5} \right\rfloor - n \left\lfloor \frac{x}{n} \right\rfloor = 1$$

at least one integer solution x with $1 \leq x \leq 200$?

Note: The symbol $\lfloor t \rfloor$ denotes the greatest integer less than or equal to the real number t .

Result. 82

Solution. When n is a multiple of 5, the left-hand side is a multiple of 5; hence, there are no solutions in that case. Furthermore, for $n = 1$, the left-hand side is always non-positive, so no solutions exist here either. In all other cases, there is always a solution if we disregard the constraint $x \leq 200$. Let us rearrange the equation to

$$5 \left\lfloor \frac{x}{5} \right\rfloor = n \left\lfloor \frac{x}{n} \right\rfloor + 1 \quad (\heartsuit)$$

for clarity; now the left-hand side is always a multiple of 5, so let us examine the solutions based on the value of $n \bmod 5$.

If $n = 5k + 4$, then $x = n + 1 = 5k + 5$ is a solution (both sides of (\heartsuit) are equal to x), so all numbers of this form are valid (40 numbers). If $n = 5k + 3$, then for the right-hand side to be a multiple of 5, we need $\lfloor x/n \rfloor$ to be at least 3, which is not possible for $n \geq 67$, as that would require a value of x to be greater than 200. For $n \leq 66$ (13 numbers) we put $x = 3n + 1$, which makes both sides of (\heartsuit) equal to x . In a similar fashion, if $n = 5k + 2$, $\lfloor x/n \rfloor$ has to be at least 2, which cannot happen for $n \geq 101$, and for the rest we pick $x = 2n + 1$ (20 numbers). Finally for $n = 5k + 1$, only $n \leq 50$ are permissible, for which $x = 4n + 1$ is used (10 numbers, but 1 does not produce a valid solution, leaving only 9 valid numbers). In total there are $40 + 13 + 20 + 9 = 82$ such numbers n .

Problem 50. Adam has an unlimited supply of 20-sided dice, each numbered from 1 to 20. He rolls a chosen number of dice at once, aiming to achieve exactly one or two ones in a single throw. What number of dice should Adam roll to maximize his probability of success?

Result. 28

Solution. The probability in question is the sum of the probability that there is exactly one one

$$P_1 = n \left(\frac{1}{20} \right) \left(\frac{19}{20} \right)^{n-1}$$

and the probability that there are exactly two ones

$$P_2 = \binom{n}{2} \left(\frac{1}{20} \right)^2 \left(\frac{19}{20} \right)^{n-2}.$$

The sum $P_1 + P_2$ can be simplified to

$$a_n = \frac{1}{2 \cdot 19^2} n(n+37) \left(\frac{19}{20} \right)^n.$$

We are interested in for which n we get $a_{n+1} < a_n$ or

$$\frac{19}{20}(n+1)(n+38) < n(n+37),$$

which further simplifies to

$$n^2 - n - 722 > 0.$$

For a positive integer n , this is equivalent to $n \geq 28$. Solving the quadratic inequality directly can be avoided by first obtaining an estimate via $n^2 > 722$ yielding $n \geq 27$, which is insufficient, but the validity of the following value is not hard to verify. This computation also shows that the sequence a_n is first increasing and then decreasing, so 28 is indeed the index of its largest term.

Problem 51. Let D be an inner point of side AC of triangle ABC such that $AD = BC$ and $BD = CD$. Furthermore, $\angle BAC = 30^\circ$. Determine the measure of angle DBA (in degrees).

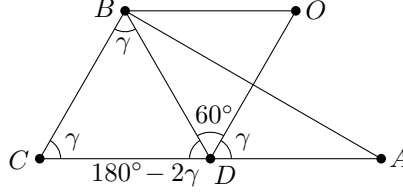
Result. 30° , 110° (2 solutions)

Solution. Let O be the circumcenter of $\triangle ABD$; then $\angle DOB = 2\angle BAD = 60^\circ$, hence $\triangle BDO$ is equilateral; in particular, $AO = DO = BD = CD$. Using $AD = BC$ we obtain congruence of isosceles triangles AOD and CDB . Denote $\gamma = \angle ACB$; then also $\angle CBD = \angle DAO = \angle ADO = \gamma$. Let us now analyze three cases based on the position of O with respect to the angle BAC .

First, assume that O is outside the angle, being closer to the ray AB than to AC . In this case, we have

$$180^\circ = \angle ADO + \angle ODB + \angle BDC = \gamma + 60^\circ + (180^\circ - 2\gamma) = 240^\circ - \gamma,$$

so $\gamma = 60^\circ$ and we easily obtain $\angle DBA = 30^\circ$.



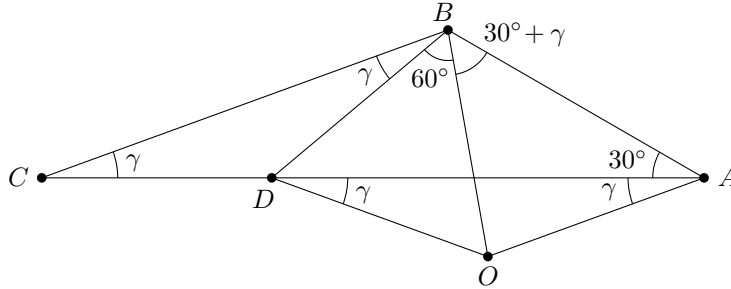
Now let O be outside $\angle BAC$ and closer to the ray AC . Then $\angle OBA = \angle BAO = 30^\circ + \gamma$, so

$$\angle CBA = \angle OBA + \angle DBO + \angle CBD = (30^\circ + \gamma) + 60^\circ + \gamma = 90^\circ + 2\gamma$$

and

$$180^\circ = \angle BAC + \angle CBA + \angle ACB = 30^\circ + (90^\circ + 2\gamma) + \gamma = 120^\circ + 3\gamma,$$

hence $\gamma = 20^\circ$. Therefore, $\angle DBA = 90^\circ + \gamma = 110^\circ$.



Finally, let us prove that O cannot be inside the angle BAC under the given conditions; this is because in such a case $\angle DOA > 120^\circ$, but $\angle BDC < 180^\circ - 60^\circ = 120^\circ$, hence the triangles AOD and BDC cannot be congruent.

Problem 52. Let f be a function that assigns a non-negative integer to each pair of non-negative integers and is defined by the following conditions:

1. For every x : $f(x, x) = 0$.
2. For every x, y : $f(x, y) = f(y, x)$.
3. For every x, y : $f(2x, 2y) = f(x, y)$.
4. For every x, y : $f(2x + 1, 2y + 1) = f(x, y)$.
5. For every x, y : $f(2x + 1, 2y) = f(x, y) + 1$.

Find the sum of all non-negative integers t , $t \leq 60$, satisfying $f(20, t) = 2$.

Result. 415

Solution. Observing the function properties, we may conclude that $f(x, y)$ counts the number of differing positions in the binary representations of x and y . Indeed, we can view it as a recursive algorithm; let $x' = \lfloor x/2 \rfloor$ and $y' = \lfloor y/2 \rfloor$, i.e., x' and y' are obtained by removing the least significant bit of x and y , respectively.

- If $x = y$, then $f(x, y) = 0$, meaning there are no differing bits.
- If both x and y are even, their least significant bits match, so we remove that bit and compute $f(x', y')$.
- If both are odd, their least significant bits still match, leading to the same reduction $f(x', y')$.
- If one is even and the other is odd, their least significant bits differ, so we increment the count by one and again proceed with $f(x', y')$.

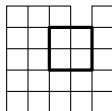
This process effectively compares the two numbers bit by bit, increasing the count precisely when corresponding bits differ.

Now, we need to find the sum of all non-negative integers $t \leq 60$ for which $f(20, t) = 2$. This means we seek numbers t with at most six binary digits that differ from $20 = 010100_2$ in exactly two positions (note that 61, 62, 63 cannot be obtained by flipping exactly two bits in the binary representation of 20). To compute the total sum, we analyze the contribution of each bit position. Since we are selecting two bits to flip out of six, there are $\binom{6}{2} = 15$ such numbers. Each bit is flipped in exactly five of these numbers (corresponding to the cases where this particular bit and one of the remaining five are flipped) and remains unchanged in the other ten numbers. We now sum these contributions accordingly.

Five numbers have a flipped least significant bit, contributing $5 \cdot 2^0$ to the total sum, while the remaining ten numbers keep this bit unchanged. Similarly, five numbers have a flipped second bit, adding $5 \cdot 2^1$ to the sum. Continuing this pattern, the contributions from the other bit positions are: $10 \cdot 2^2$ (as this bit contributes in the ten non-flipped cases), $5 \cdot 2^3$, $10 \cdot 2^4$, and $5 \cdot 2^5$. It follows easily that the total sum is given by

$$5 \cdot (010100_2 + 111111_2) = 5 \cdot (20 + 63) = 415.$$

Problem 53. Becky drew a 45×45 grid and counted the 1×1 squares within it, realizing there were 2025. This made her unhappy, as she prefers figures with 2024 small squares for personal reasons. To fix this, she removed one 1×1 square from the boundary of the grid. Afterwards, she counted all possible squares (not necessarily 1×1) in the adjusted grid. Since Becky is superstitious and afraid of numbers divisible by 13, she cut off a square such that the total number of squares in the adjusted grid is not divisible by 13. The diagram below illustrates an example: a 5×5 grid with one boundary square removed and a 2×2 square in the adjusted grid. Determine how many possible boundary squares Becky could remove to meet her condition.



Result. 152

Solution. The total number of squares in the original 45×45 grid is given by

$$S = 1^2 + 2^2 + \cdots + 45^2 = \frac{1}{6} \cdot 45 \cdot 46 \cdot 91.$$

When Becky removes a single 1×1 square from the boundary, the total count of squares decreases by R , the number of squares that contained the removed square. Since S is itself divisible by 13, the adjusted total will be divisible by 13 if and only if R is divisible by 13. Thus, we must find values of R that are multiples of 13.

To determine R , note that each square X containing a given boundary square x is uniquely determined by selecting two corner squares of X along the boundary such that x lies between them (either corner square may coincide with x). Hence, for a boundary square located at position n along one side (counting from a corner), the number of such squares is:

$$R = n(46 - n).$$

Thus, we need to identify values of n such that $n(46 - n)$ is divisible by 13, as these positions must be avoided. Checking values for $1 \leq n \leq 23$ (as symmetry allows us to consider only half of one side), we find that divisibility by 13 occurs precisely for $n = 7, 13, 20$. Therefore, each side of the grid has six such boundary squares, none of them being a corner, so across the four sides, there are $4 \cdot 6 = 24$ boundary squares that must be avoided. The total number of boundary squares is $4 \cdot 44 = 176$. Thus, the number of valid choices for Becky is $176 - 24 = 152$.

Problem 54. Hare and Tortoise are competing in a race. The Tortoise goes slowly but steadily, while the Hare runs 6 times faster, but every time it runs 9 meters forward, it comes back 7 meters to mock the Tortoise. Consider the time interval from the start of the race until the last moment they meet on the track. What fraction of that time was the Tortoise in the lead?

Result. $\frac{22}{45}$

Solution. We analyze the (signed) difference in position between the Hare and the Tortoise, where a positive value indicates the Hare is ahead. The Hare runs forward 9 meters but then moves backward 7 meters, effectively gaining $9 - \frac{9}{6} = \frac{45}{6}$ meters forward and losing $7 + \frac{7}{6} = \frac{49}{6}$ meters backward. Since we only need the time ratio, we scale all distances by 6 for simpler calculations. To further simplify the analysis, we shift to a frame of reference in which the Tortoise is stationary. In this frame, the Hare moves 45 meters forward and 49 meters backward in each of its cycles.

In each cycle, the Hare spends a certain distance behind the Tortoise: 4 meters in the first cycle, 12 meters in the second, and so on. Each cycle covers 94 meters, so the last full cycle is the eleventh, where the Hare spends 84 meters

behind and finishes 44 meters behind the Tortoise. After this, the Hare runs 46 more meters until it meets the Tortoise for the last time, with 44 of those meters spent trailing behind.

The total distance covered by the Hare until this final meeting is $11 \cdot 94 + 46 = 1080$ meters. Out of this, the distance spent trailing the Tortoise is $(4 + 12 + \dots + 84) + 44 = 528$ meters. Thus, the fraction of time the Tortoise was in the lead until their last meeting is $\frac{528}{1080} = \frac{22}{45}$.

Problem 55. Mark received a sequence $\{a_n\}_{n=1}^{\infty}$ as a gift, defined by the initial terms $a_1 = 1$, $a_2 = 3$, and the recurrence relation

$$a_{n+1}^2 + 3a_n^2 - 4a_{n-1}^2 = 4a_n \cdot (a_{n+1} - a_{n-1}) + 2n - 1$$

for all $n \geq 2$. However, the sequence is not uniquely determined by this relation. To resolve any ambiguity, Mark calculated the terms step by step, always choosing the largest value whenever more than one possibility arose. Determine the value of a_{13} .

Result. 12274

Solution. Rearranging the recurrence relation, we obtain

$$a_{n+1}^2 + 4a_n^2 - a_n^2 - 4a_{n-1}^2 - 4a_n \cdot a_{n+1} + 4a_n \cdot a_{n-1} = 2n - 1,$$

which simplifies to

$$(a_{n+1} - 2 \cdot a_n)^2 - (a_n - 2 \cdot a_{n-1})^2 = 2n - 1.$$

Since $(a_2 - 2 \cdot a_1)^2 = (3 - 2)^2 = 1 = 1^2$ and $(n - 1)^2 + 2n - 1 = n^2$, an easy inductive argument shows that

$$(a_{n+1} - 2 \cdot a_n)^2 = (a_n - 2 \cdot a_{n-1})^2 + 2n - 1 = n^2.$$

This yields two candidate values of the term a_{n+1} :

$$a_{n+1} = 2 \cdot a_n + n \quad \text{or} \quad a_{n+1} = 2 \cdot a_n - n.$$

Since Mark always selects the larger value, he consistently chooses

$$a_{n+1} = 2 \cdot a_n + n.$$

Using this recurrence iteratively from $a_1 = 1$, we get

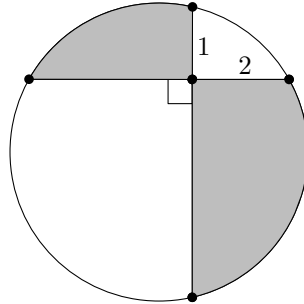
$$a_{13} = 1 \cdot 2^{12} + 1 \cdot 2^{11} + 2 \cdot 2^{10} + 3 \cdot 2^9 + \dots + 12 \cdot 2^0.$$

This can be simplified in the following manner:

$$\begin{aligned} a_{13} &= 1 \cdot 2^{12} + 1 \cdot 2^{11} + 2 \cdot 2^{10} + 3 \cdot 2^9 + \dots + 12 \cdot 2^0 \\ &= 2^{12} + (2^{11} + 2^{10} + \dots + 2^0) + (2^{10} + 2^9 + \dots + 2^0) + (2^9 + 2^8 + \dots + 2^0) + \dots + (2^1 + 2^0) + 2^0 \\ &= 2^{12} + (2^{12} - 1) + (2^{11} - 1) + \dots + (2^2 - 1) + (2^1 - 1) \\ &= 2^{12} + (2^{13} - 1) - 13 \\ &= 4096 + 8192 - 14 = 12274. \end{aligned}$$

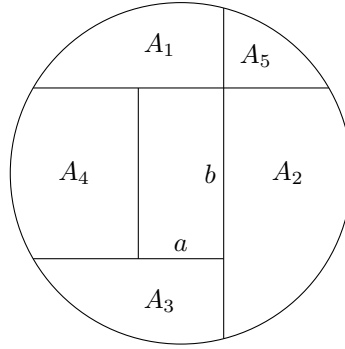
Therefore, Mark's value of a_{13} is 12274.

Problem 56. The diagram shows a circle divided by two perpendicular chords. Two segment lengths are provided (both shorter than the remaining part of the respective chord), along with the information that the ratio of the gray area to the white area is $\frac{5\pi-2}{5\pi+2}$. Determine the radius of the circle.



Result. $\frac{5\sqrt{2}}{2}$

Solution. Consider the reflection of the two chords through the center of the circle. Let us denote some of the areas and lengths as in the diagram.



Clearly, $A_1 = A_3$ and $A_2 = A_4 + A_5$. This means that the area G of the gray region is equal to

$$G = A_1 + A_2 = A_3 + A_4 + A_5,$$

while the area of the white region is

$$W = A_3 + A_4 + A_5 + ab = G + ab.$$

Taking into account the known ratio of the areas,

$$\frac{G}{W} = \frac{G}{G + ab} = \frac{5\pi - 2}{5\pi + 2},$$

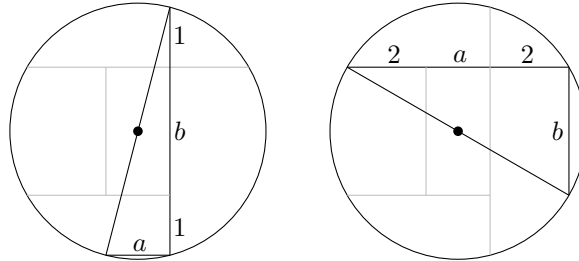
which can be rearranged to

$$G = \frac{1}{4}(5\pi - 2)ab.$$

Let us now consider the area of the whole circle, whose radius we denote by r :

$$\pi r^2 = G + W = 2G + ab = \frac{5\pi}{2}ab$$

or $2r^2 = 5ab$. We obtain another two equations by observing that we can rearrange the segments in two ways to obtain a right-angled triangle inscribed to the circle: One with legs a and $b + 2$, one with $a + 4$ and b .



So we are left with a system of three equations

$$\begin{aligned} 2r^2 &= 5ab, \\ 4r^2 &= a^2 + (2 + b)^2, \\ 4r^2 &= (4 + a)^2 + b^2, \end{aligned}$$

which can be easily solved – comparing the last two equations one gets $b = 2a + 3$, which can be plugged into the first two equations to get a quadratic equation in a . One of the solutions ($a = -\frac{5}{3}$) does not make sense in the present situation, the other one leads to $a = 1$, $b = 5$, and $r = \frac{5\sqrt{2}}{2}$.

Problem 57. A building has 160 floors. Each floor's hallway is accessible via either of two main doors, and the hallway contains four rooms. Each room has its own door, with one person living in each room. A single lock will be installed on every door, including the hallway doors, and keys will be distributed to ensure that:

- Each person can access their own room but not anyone else's,
- Each person can access the hallway on their floor, and

- It is allowed for someone to have access to hallways on other floors.

Each lock is associated with a unique corresponding key and can only be opened by that key. However, the same lock can be used on multiple doors if needed, and any number of copies of its corresponding key can be made and distributed. Individuals can hold as many keys as required. The company responsible for the installation aims to minimize the total cost of producing the keys. Creating a new key costs 3, and making a copy of an existing key costs 2. What is the minimal total cost of all keys while meeting the above conditions?

Result. 2432

Solution. To minimize costs, we note that if a resident has a single unique key that unlocks both their room and one hallway door, it is cheaper than having two separate keys (even if they were just copies). However, this setup means that one hallway door must remain inaccessible to others, as sharing access to it would also grant access to the resident's room.

Thus, the problem reduces to minimizing the key cost for 160 floors where each hallway has only one entrance and three separate rooms. Let there be n distinct key types $1, 2, \dots, n$. Each resident receives one key for their room and one key for the hallway. These two keys cannot be of the same type; indeed, if a resident's room key was the same as their hallway key, the other two residents sharing the hallway key would either gain access to that resident's room or be entirely unable to enter the hallway. Also, no two residents can have the same pair of key types, as this would mean they have access to the same room. Therefore, the number of valid key pairings is given by $\frac{1}{2}n(n-1)$.

Since we are distributing a total of $3 \cdot 160 = 480$ key pairs, we determine the lower bound for n from the inequality

$$\frac{1}{2}n(n-1) \geq 480.$$

Solving for a positive integer n , we find that $n \geq 32$. Now, we demonstrate that it is possible to use exactly $n = 32$ distinct key types. To achieve this, we divide the 160 floors into 32 groups of 5 floors each. The key assignments proceed as follows (the first key in the pair is for the hallway and the second for the room):

- The 15 residents in the first group receive the key pairs $(1, 2), (1, 3), \dots, (1, 16)$.
- The second group is assigned key pairs $(2, 3), (2, 4), \dots, (2, 17)$.
- This pattern continues, with the 31st group receiving key pairs $(31, 32), (31, 1), \dots, (31, 14)$.
- Finally, the last group (32nd) is assigned key pairs $(32, 1), (32, 2), \dots, (32, 15)$.

It is easy to see that this distribution satisfies the conditions given above. Therefore, $n = 32$ is the minimal number of unique keys required, to which we add 928 copies to cover all the $160 \cdot 3 \cdot 2 = 960$ keys.

Accounting for the residents mentioned in the first paragraph, a total of $32 + 160 = 192$ unique keys must be produced. Thus, the final cost is calculated as $192 \cdot 3 + 928 \cdot 2 = 2432$.