Problem 1. If the arithmetic mean of four distinct positive integers is equal to 10, what is the largest possible value of any of these integers?

Result. 34

Solution. To have one of the numbers as large as possible, the rest has to be as small as possible. Since the numbers are distinct, the least three possible values are 1, 2, and 3. For the average to be equal to 10, i.e. the sum to be equal to $4 \cdot 10 = 40$, the remaining number has to be $40 - (1 + 2 + 3) = 34$.

Problem 2. If 4 is a solution of the quadratic equation $x^2 + mx + 2020 = 0$ with an integer m, what is the other solution?

Result. 505

Solution. Since 4 is a root, we get $4^2 + 4m + 2020 = 0$ or $m = -509$, so the equation now reads $x^2 - 509x + 2020 = 0$ with solutions 4 and 505.

Alternatively, if s is the other solution of the given equation, then

$$x^2 + mx + 2020 = (x - 4)(x - s) = x^2 - 4x - sx + 4s$$

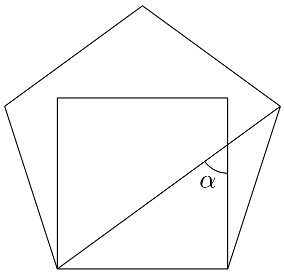
and comparing the coefficients of the polynomials yields $4s = 2020$ or $s = 505$.

Problem 3. Number 95 gives 4 as the remainder after division by a positive integer N. What is the least possible value of N?

Result. 7

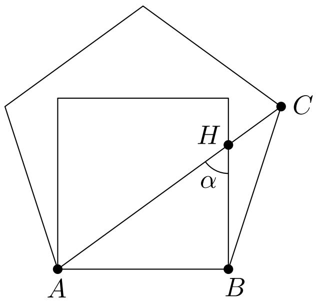
Solution. As N is larger than 1 and divides $95 - 4 = 91 = 7 \cdot 13$, the smallest possible value is 7.

Problem 4. A square and a regular pentagon are as in the picture below. Find the angle α in degrees.



Result. 54°

Solution. Let us label the points as in the picture.



The size of the interior angle in a regular pentagon is 108°. The triangle ABC is isosceles with $\angle ABC = 108°$, so

$$\angle BAH = \angle BAC = \tfrac{1}{2}(180° - 108°) = 36°.$$

Since ABH is a right-angled triangle with the right angle at vertex B,

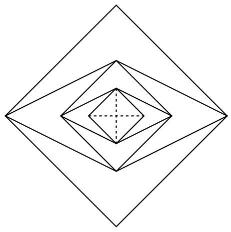
$$\alpha = \angle AHB = 180° - 90° - 36° = 54°.$$

Problem 5. A bus stop is served by three bus lines A, B, and C, which leave the stop in intervals of 12, 10, and 8 minutes, respectively. When Brian walked past the stop, he noticed that the three buses of the three lines had left the stop simultaneously. After how many minutes from that point will that happen again for the first time?

Result. 120

Solution. The sought number of minutes has to be a multiple of all three periods, and since we are looking for the least such number, the answer is the least common multiple of 12, 10, and 8, which is 120.

Problem 6. The rhombus flower grows according to the following pattern: In the middle there is a square blossom with two diagonals of length 1. In the first step the horizontal diagonal is doubled creating a new quadrilateral. In the next step the vertical diagonal is doubled and again a new quadrilateral blossom is generated. This procedure is continued until there is a flower with five quadrilateral blossoms. Find the perimeter of the outer (i.e. the fifth) blossom.



Result. $8\sqrt{2}$

Solution. The fifth blossom is a square with diagonals of length 4, hence the length of its side is $2\sqrt{2}$ and the perimeter equals $8\sqrt{2}$.

Problem 7. A botanist planted two plants, P_1 and P_2, of the same species and measured their heights. After a week, during which the plants had grown up by the same percentage, he measured again and noticed that P_1 was as big as P_2 had been a week before and P_2 was by 44% bigger than P_1 had been a week before. By what percentage did the plants grow during the week?

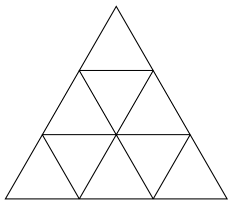
Result. 20%

Solution. Let us denote by P_1 and P_2 the original heights of the plants. Since they both grew by the same percentage during the week, their new heights are kP_1 and kP_2, respectively, for some real number $k > 1$ such that $(k-1) \cdot 100\%$ is the sought percentage. Then the measurements imply

$$\begin{aligned} kP_1 &= P_2, \\ \frac{kP_2}{P_1} &= 1.44. \end{aligned}$$

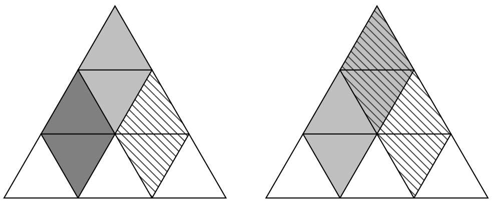
Substituting for P_2 in the second equation and cancelling P_1 in the fraction yields $k^2 = 1.44$, so $k = 1.2$, meaning that the plants grew up by 20%.

Problem 8. How many parallelograms are there in the picture?



Result. 15

Solution. For each of the three vertices of the large triangle, there are three rhombi "pointing" in direction of the vertex and two 1×2 parallelograms sharing that vertex with the triangle. The picture shows these two types of parallelograms for the top vertex.



No other types of parallelograms are present in the picture, so in total there are $3 \cdot (2 + 3) = 15$ parallelograms.

Problem 9. A bus company offers buses for 27 or 36 passengers. A tour group consisting of 505 tourists wants to travel with buses of that company. These buses have been selected by the company so that the total number N of empty seats in the buses is as small as possible. Determine N.

Result. 8

Solution. We are looking for the smallest number $s \geq 505$ of the form $s = 27x + 36y$, where x and y stand for the numbers of buses of the first and the second type, respectively. Since the greatest common divisor of 27 and 36 is 9, s has to be a multiple of 9. The smallest multiple of 9 which is greater or equal to 505 is 513 and since $513 = 27 \cdot 3 + 36 \cdot 12$, we conclude that the number of empty seats is $513 - 505 = 8$.

Problem 10. A fan of animals bought two identical pictures of a wolf and four identical pictures of a fox. He wants to hang them next to each other on six nails on a wall in his living room. Moreover, he wants to change their order every day in such a way that the resulting sequence looks different from the sequences in all the preceding days. Finally, he does not want the two pictures of a wolf to hang next to each other. What is the highest number of days for which he can do it?

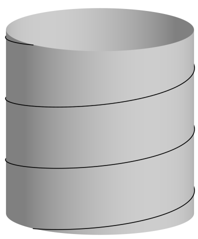
Result. 10

Solution. In other words, we ask for the number of distinct sequences of the pictures not having the two wolves next to each other. The left-hand wolf can be hung on positions 1, 2, 3, and 4 out of 6, and for each of these positions, the right-hand wolf can be hung to the following positions:

$$\begin{aligned} &1 : 3, 4, 5, 6 \\ &2 : 4, 5, 6 \\ &3 : 5, 6 \\ &4 : 6, \end{aligned}$$

so there are 10 such sequences in total.

Problem 11. A cylinder of height 18 cm and circumference 8 cm has a string tightly wound around it three times, starting at the bottom of the cylinder and ending at the top in the point precisely above the starting point. What is the length of the string in cm?



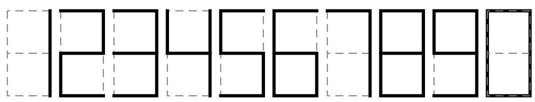
Result. 30

Solution. Unfolding the cylinder, we see that during each turn around the cylinder, the string has risen by 6 cm while advancing by the circumference 8 cm in the horizontal direction. By the Pythagorean theorem, the segment of the string corresponding to one turn is

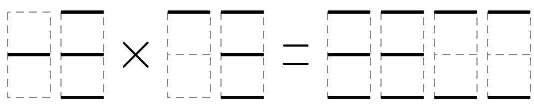
$$\sqrt{6^2 + 8^2} = 10\,\text{cm}$$

long. Since there are three turns, we conclude that the total length of the string is 30 cm.

Problem 12. A correctly working calculator displays digits in the following way:



Adam's calculator fell out of a window and now it shows only the horizontal segments. To verify that the calculator still computes correctly, Adam performed the following calculation:



What is the sum of all digits appearing in this calculation?

Result. 33

Solution. The last two digits must be zeroes. Furthermore, the first digit of the first factor is 4 and the first digit of the second factor is 7. Since the product is divisible by 100 and consequently by 25, one of the factors has to be divisible by 25 or both factors by 5. There is no two-digit multiple of 25 starting with 4, hence the second factor has to be divisible by 5 and since it cannot end with 0, it equals 75. Moreover, since the product is divisible by 4, the first factor must be divisible by 4 and so it is 48. We have $48 \times 75 = 3600$, where the sum of all digits present is 33.

Problem 13. Ed had to add up two numbers, but he accidentally wrote an additional digit at the end of one number. As a result he got the sum of 44444 instead of 12345. What was the smaller of the two numbers that Ed originally wanted to add up?

Result. 3566

Solution. Let x and y be the two numbers and c a digit that Ed wrote to (say) the number x. Then we have

$$x + y = 12345 \quad \text{and} \quad (10x + c) + y = 44444,$$

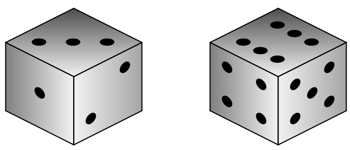
therefore

$$9x + c = 32099.$$

It follows that c has to be 5 since $32099 - c$ has to be divisible by 9. We conclude that $x = 3566$ and $y = 8779$.

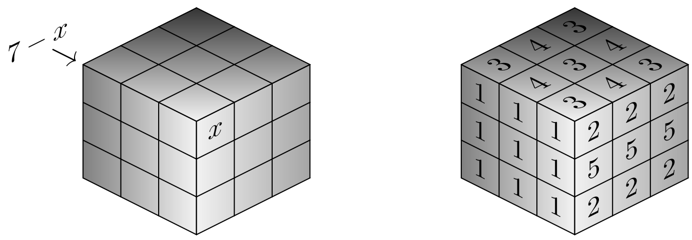
Problem 14. Peter is given 27 standard dice and asked to glue them together into a larger $3 \times 3 \times 3$ cube, so that the adjacent faces (i.e. the faces glued together) have the same number of dots. What is the maximal number of dots that Peter can leave visible on the outside of his $3 \times 3 \times 3$ cube?

Note: Two views of the standard dice are shown below. The faces are arranged so that opposite sides add to 7.

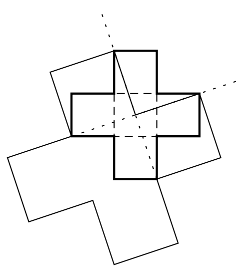


Result. 189

Solution. The key observation is that the dots on opposite faces on the large cube always add up to 7, as shown in the left-hand figure below. The large cube has 27 pairs of opposite faces, so no matter how Peter arranges his dice, the total number of dots showing must be $27 \cdot 7 = 189$. One possible arrangement Peter can use is shown in the right-hand figure below.

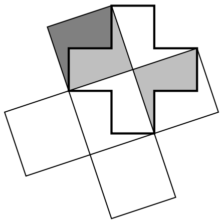


Problem 15. Antonia drew a small X-pentomino made of 5 congruent squares. Then she drew two perpendicular diagonals of this pentomino with dotted lines. Finally she constructed a bigger X-pentomino with some of the sides lying on the diagonals of the small pentomino as in the figure. Find the ratio of the area of Antonia's big pentomino to the area of the small one.



Result. $5:2$.

Solution. The diagonals divide the small X-pentomino into four congruent pieces. Furthermore, two such pieces can be glued together to form one square of the larger pentomino, as the picture shows. Thus the large pentomino can be divided into ten such pieces and the sought ratio of areas is $10 : 4 = 5 : 2$.



Problem 16. How many palindromes between 10^3 and 10^7 have an even sum of digits?

Note: A *palindrome* is a number which stays the same when the order of its digits is reversed, e.g. 12321 is a palindrome.

Result. 5940

Solution. All numbers between 10^3 and 10^4 have an even number of digits, hence the sum of digits of every such palindrome is even. Moreover, the palindromes from this range are exactly the numbers of the form $\overline{abba}$, where a, b are any digits, a non-zero, which shows that there are 90 palindromes in that range. In a similar way we infer that there are precisely 900 palindromes between 10^5 and 10^6 and the sum of digits of each of them is even.

The palindromes between 10^4 and 10^5 are of the form $\overline{abcba}$ for a, b, c digits, a non-zero, and the sum of their digits is clearly even if and only if c is even. Hence there are 9 options for a, 10 for b, and 5 for c, which gives $9 \cdot 10 \cdot 5 = 450$ sought palindromes in the given range. A similar argument applies to the range from 10^6 to 10^7, showing that it contains 4500 palindromes the sum of whose digits is even.

We conclude that out of all palindromes between 10^3 and 10^7, $90 + 900 + 450 + 4500 = 5940$ have an even sum of digits.

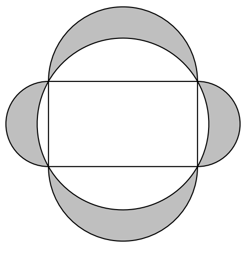
Problem 17. A women's choir consists of sixty singers: twenty sopranos, twenty mezzo-sopranos, and twenty altos. Moreover, six singers in each voice are very skilled, so they are able to sing any of the parts if necessary; the rest of the singers can sing their part only. What is the highest number S such that whenever S of the singers fall sick and cannot sing, the remaining singers can rearrange themselves to form a choir with at least ten singers in each voice?

Result. 22

Solution. If, for example, all the altos together with three other "skilled" singers fall sick, then there are only nine skilled singer remaining, so there is no way to have ten alto singers. Therefore $S < 23$.

On the other hand, observe that if some of the singers fall sick, the situation can only get worse if the sick ones are the "skilled" ones instead of the "ordinary" ones. Therefore, if 22 singers fall sick, we can assume that 18 of them are the skilled ones, so only four "ordinary" singers fall sick and it is immediate that at least ten singers remain in each voice. This shows that $S \geq 22$ and consequently, $S = 22$.

Problem 18. A rectangle with sides of length 3 and 4 is inscribed into a circle. Moreover, four half-circles are glued to its sides from outside as in the picture. What is the area of the shaded region, which consists of points of the half-circles not lying inside the circle?



Result. 12

Solution. Using the Pythagorean theorem, we find that the length of a diagonal of the rectangle is

$$\sqrt{3^2 + 4^2} = 5.$$

This diagonal is the diameter of the circumscribed circle, the area of which therefore equals $\pi(5/2)^2$. Moreover, the outer half-circles have radii $4/2$ and $3/2$, respectively, so the total area of the semicircles is

$$2 \cdot \tfrac{1}{2}\pi \left(\tfrac{3}{2}\right)^2 + 2 \cdot \tfrac{1}{2}\pi \left(\tfrac{4}{2}\right)^2 = \pi \left(\tfrac{5}{2}\right)^2.$$

Consider the whole region formed by the semi-circles and the rectangle; its area is

$$12 + \pi \left(\tfrac{5}{2}\right)^2 .$$

The shaded region is obtained by removing the circumscribed circle from this all-encompassing region, so its area equals 12.

Alternatively, we can argue as follows. The Pythagorean theorem gives relation of squares raised above three sides of a right triangle. The same relation holds for half-circles, which we apply to both halves of the given rectangle (divided by a diagonal). Straightforward observation then gives that any two neighbouring grey areas have the sum of half of the rectangle. Therefore the total grey area is equal to that of the rectangle, which is $3 \cdot 4 = 12$.

Problem 19. Find the largest three-digit prime number p_1 such that the sum of all digits of p_1 is a two-digit prime p_2 and the sum of the digits of p_2 is a one-digit prime p_3.

Result. 977

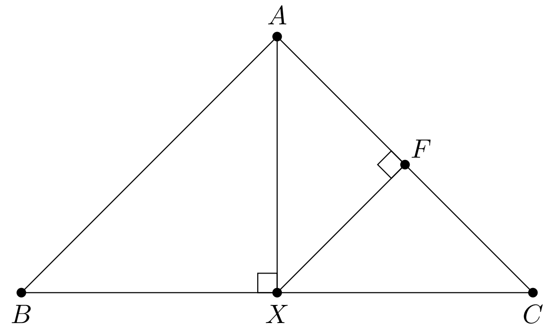
Solution. The sum of the digits of a three-digit number is at most $9 + 9 + 9 = 27$. There are five two-digit primes not greater than 27, namely 11, 13, 17, 19, and 23. The sums of the digits of these primes are 2, 4, 8, 10, 5, respectively. Therefore, $p_2 = 11$ or $p_2 = 23$ are the only possibilities. The largest three-digit prime number with the digit sum of 23 is 977. Since $977 > 911$, which is the largest three-digit prime having its digit sum equal to 11, the sought number is 977.

Problem 20. In triangle ABC satisfying $AB = AC$, there is a side axis which meets one of the altitudes in a single point lying on the perimeter of ABC. Determine all possible sizes of angle ACB in degrees.

Result. $45°$, $67.5°$

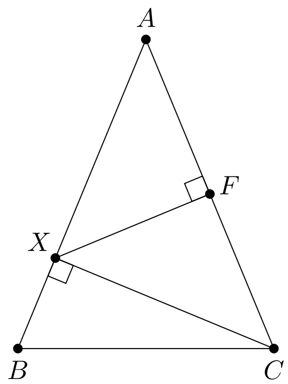
Solution. Let us denote the intersection point by X and the centre of AC by F. Observe that the altitude going through X must be the one corresponding to the same side as X belongs to. We now examine all possible locations of X.

If X lies on the base BC, it must be the intersection of the altitude from A and one of the side axes. From symmetry of triangle ABC we infer that this altitude coincides with the axis of BC, so the single-point intersection has to be with one of the remaining axes. The symmetry then implies that it actually intersects the axes of both AB and AC in a single point.



Since FX is the axis of AC, triangle AXC is isosceles. Moreover, we have $\angle AXC = 90°$ and so $\angle ACB = \angle ACX = 45°$.

If X lies on AB, then it must be the intersection of the altitude from C and the axis of AC.



As in the previous case, triangle AXC is isosceles and right-angled with right angle at X, implying that $\angle BAC = \angle XAC = 45°$. Therefore,

$$\angle ACB = \tfrac{1}{2}(180° - \angle BAC) = 67.5°$$

in this case. The case when X lies on AC is completely symmetrical.

Problem 21. Find the sum of all positive divisors of 3599.

Note: Divisors include 1 and 3599.

Result. 3720

Solution. We have

$$3599 = 3600 - 1 = 60^2 - 1 = (60+1)(60-1) = 59 \cdot 61.$$

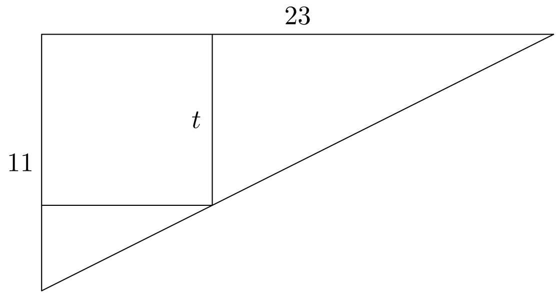
It is easy to see that both 59 and 61 are prime numbers, therefore the answer is $1 + 59 + 61 + 3599 = 3720$.

Problem 22. Sisters Dolly, Holly, and Molly made a campfire and roasted sausages. Dolly bought 17 sausages, Holly bought 11, and Molly bought none. When they had eaten all of them, they decided to share the costs equally. How much money should Dolly get, if Molly paid $28 to her sisters to get rid of her debt?

Result. 23

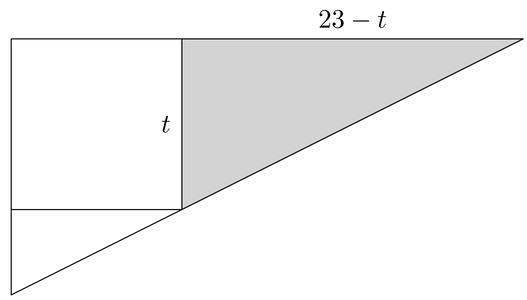
Solution. Molly paid exactly one third of the total costs, that is, all the sausages together costed $28 \cdot 3 = 84$ dollars. Dolly paid for 17 sausages out of 28, that is $84 \cdot 17/28 = 51$ dollars. Since the sisters decided to share the costs equally, she should have paid only 28 dollars. Thus she should get $51 - 28 = 23$ dollars.

Problem 23. The legs of a right-angled triangle have lengths 11 and 23. A square of side length t has two of its sides lying on the legs of the triangle and one vertex on its hypotenuse as in the picture. Find t.



Result. $\frac{253}{34}$

Solution. The gray triangle in the picture is right-angled and shares one angle with the large triangle, hence these two triangles are similar.



The ratio of lengths of the legs has to be the same for the two triangles, thus we obtain the equation

$$\frac{23-t}{t} = \frac{23}{11}$$

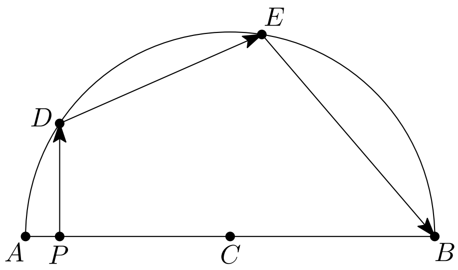
with the solution $t = 253/34$.

Problem 24. Find the smallest positive integer n for which $11 \cdot 19 \cdot n$ is equal to a product of three consecutive integers.

Result. 840

Solution. As 11 and 19 are primes, one of the three consecutive numbers has to be divisible by 11 and one, not necessarily a different one, by 19. Moreover, since the product is positive, all three numbers have to be positive as well. That is, we search for small positive multiples of 11 and 19 differing by at most 2. The smallest are $3 \cdot 19 = 57$ and $5 \cdot 11 = 55$ so we only have to supplement the product by 56 to get $55 \cdot 56 \cdot 57 = 11 \cdot 19 \cdot 840$. Therefore, 840 is the sought number.

Problem 25. Consider a semi-circle with centre C and diameter AB. A point P on AB satisfies the following. A laser beam leaves P in a direction perpendicular to AB, bounces off the semicircle at points D and E following the rule of reflection, that is, $\angle PDC = \angle EDC$ and $\angle DEC = \angle BEC$, and then it hits the point B. Determine $\angle DCP$ in degrees.



Result. $36°$

Solution. Let us denote $\angle DCP$ by x. Since D and E both lie on the circle with center C, $\triangle DCE$ is isosceles with base DE. The first given equality $\angle PDC = \angle EDC$ implies that $\triangle CDP$ is congruent to a half of $\triangle CDE$ (in particular to $\triangle CDM$ where M is midpoint of DE). Using also the second equality it follows that $\angle BCE = \angle ECD = 2\angle DCP (= 2x)$ and as these three angles form together a straight angle we have $x + 2x + 2x = 180° \Rightarrow x = \angle DCP = 36°$.

Problem 26. There are 2020 towns labelled $1, 2, 3, \ldots, 2020$ in a country. The president decided to build a railway network. To save money, he built tracks only between the pairs of cities labelled a and b, $a < b$, which satisfied the following condition: Number b is a multiple of a and there is no c such that $a < c < b$, c is a multiple of a, and b is a multiple of c. With how many other cities is the city labelled 42 connected?

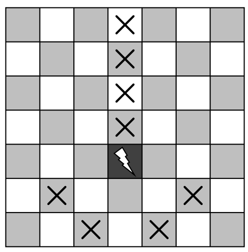
Result. 18

Solution. Consider a pair of connected cities. One of them must have one prime more in their prime factorisation. They cannot have the same primes because that would mean the cities would be the same. On the other hand, they cannot have two or more primes difference. To prove that, let p, q be not necessarily distinct primes and let cities a and $b = a \cdot p \cdot q$ be connected. Than $a \cdot p$ divides b, thus it violates the second condition.

City with number 42 has prime factorisation $2 \cdot 3 \cdot 7$. That means that cities with lower index number connected to this city are three, with these prime factorisations: $2 \cdot 3$, $2 \cdot 7$ and $3 \cdot 7$.

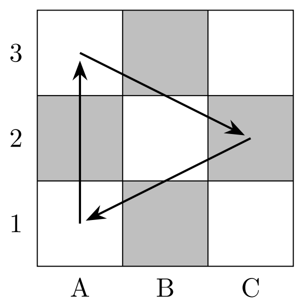
The indexes greater than 42 could be written in the form $42 \cdot p$ for some prime p. The largest such prime satisfying $42 \cdot p < 2020$ equals 47, so $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$. This means there are fifteen more cities connected to the city labelled 42, that is, eighteen in total.

Problem 27. Marek invented a chess piece called the *blitzer*. The blitzer can move forward like a rook and backward like a knight just as shown in the picture. Marek placed it in the middle square of a different chessboard of dimensions 3×3 and moved the blitzer 2020 times according to the rules. At most how many times could the blitzer have visited a single square? The initial position of the blitzer does not count as a visit.



Result. 673

Solution. It is clear from the rules that if the blitzer moves from one square to another, it cannot return to the starting square in the following move. However, it can return to that square in two moves, even on a 3×3 chessboard, as the picture shows:



Therefore if the blitzer gets to one of the squares A1, A3, C2, it can start "cycling" and reach visit each of these squares in its every third move.

If the blitzer starts on B2, then its only possible first move is a rook move to B3 and the subsequent options are A1 or C1 via a knight move. Therefore A1 is reachable from B2 using two moves, but the blitzer cannot directly move to B2 from A1 or C1 to have the cycle of three moves sooner, so we have to use at least two moves to get in such a cycle. That leaves 2018 moves for cycling the blitzer and since $2018 = 672 \cdot 3 + 2$, the blitzer can do 672 cycles. Thus, it visits A1 673 times.

Problem 28. David raced against a snail on a circuit with start and finish at the same point. They started at the same time, ran in the same direction and met in the finish. However, the snail was faster, so it completed more laps than David, who completed only three laps, and therefore the two met 2020 times, including the meeting at the start and in the finish. The following day, they ran the same race, but David changed the direction in which he ran. Their speeds were the same as the day before. How many times did they meet in the second race?

Result. 2026

Solution. Let us assume that the snail completed exactly n laps during the race. That is, the speed of David was 3 and that of snail n (in laps per race). Then the rate of change of the margin, i.e. the oriented distance from David to the snail, was $n-3$. Since they met whenever the margin was a natural number, they met $n-3$ times, excluding the beginning. Thus $n = 2022$. When they were running in opposite directions, the rate of change of their oriented distance was $n+3 = 2025$, so they met 2025 times excluding the start, i.e. 2026 times in total.

Problem 29. Noah plays a game, in which his character collects three types of items: support, attack and defence. Each of them can be of a level from 1 to 10. It is possible to combine two different items of the same level to obtain an item of the third type of one higher level. For example, combining a defence item of level 3 with a support item of level 3 results in an attack item of level 4. How many attack items of level 1 must Noah collect in order to obtain an attack item of level 10 provided he has an unlimited supply of defence and support items of level 1?

Result. 170

Solution. Let us denote s_i, d_i, a_i number of support, defence and attack items of the level i needed to get one attack item of level 10. We have $s_{10} = d_{10} = 0$, $a_{10} = 1$ and according to the rules,

$$\begin{aligned} s_{i-1} &= d_i + a_i, \\ d_{i-1} &= a_i + s_i, \\ a_{i-1} &= s_i + d_i \end{aligned}$$

for all $i \in \{2, \ldots, 10\}$. Using these rules, one can fill in a 10×3 table so that with the exception of the top row $(0, 0, 1)$, value in every cell is the sum of the two numbers in the row above it, but not in the same column:

0	0	1
1	1	0
1	1	2
3	3	2
5	5	6
11	11	10
21	21	22
43	43	42
85	85	86
171	171	170

Thus the answer is 170.

An alternative approach is to note that since the roles of support and defence items are interchangeable, actually $s_i = d_i$. It is also easy to prove by induction that $|a_i - s_i| = 1$; this holds for $i = 10$ and

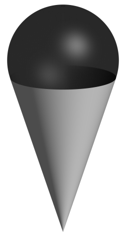
$$|a_{i-1} - s_{i-1}| = |(s_i + d_i) - (a_i + d_i)| = |s_i - a_i| = 1.$$

Finally, we have

$$s_{i-1} + d_{i-1} + a_{i-1} = (d_i + a_i) + (a_i + s_i) + (s_i + d_i) = 2(s_i + d_i + a_i),$$

so $s_1 + d_1 + a_1 = 2^9 = 512$. These observations together imply that $s_1 = d_1 = 171$ and $a_1 = 170$.

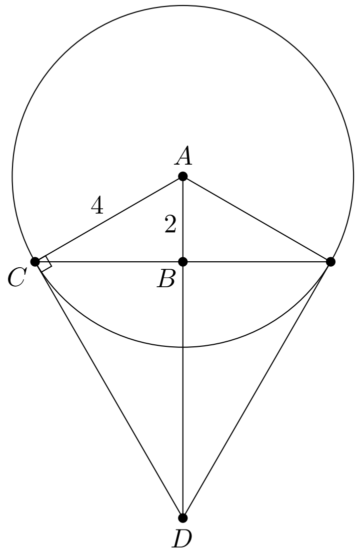
Problem 30. Giuseppe bought an ice cream. It had the shape of a ball of radius 4 cm in an ice cream cone. Giuseppe noticed that the ice cream ball fitted in the cone in the following way: The centre of the ball was precisely 2 cm above the base of the cone and the cone surface ended exactly where it touched the ball tangentially. What was the volume of the cone?



Result. 24π

Solution. Let $AC = 4$ be the radius of the ball, BC the radius of the cone base and BD the cone height as in the figure below. From the Pythagorean theorem applied to the triangle ABC we get $BC^2 = AC^2 - AB^2 = 16 - 4 = 12$. From similarity of triangles ABC and CBD we can deduce $\frac{BD}{BC} = \frac{BC}{AB}$. Therefore the volume is

$$V = \frac{1}{3}\pi BC^2 \cdot BD = \frac{1}{3}\pi BC^2 \frac{BC^2}{AB} = \frac{\pi \cdot 12^2}{3 \cdot 2} = 24\pi.$$



Problem 31. Using each of the digits 1, 2, 3, 4, 6, 7, 8, and 9 exactly once, Bob formed two four-digit numbers, which he subsequently added together. Find the highest possible sum of the digits of the result.

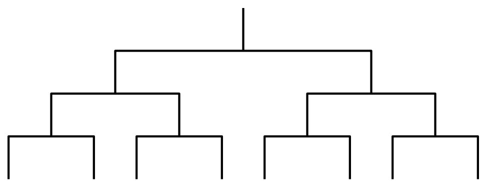
Result. 31

Solution. Let us first recall the basic property of addition: We add the numbers digit-by-digit with the exception that carries must added properly. That is, we add the four pairs of digits and the carries. Since we are adding only two numbers, the carries are at most 1.

Let us now denote the two numbers by a and b and the sum of digits of any number n by $S(n)$. Then it holds that $S(a + b) = S(a) + S(b) - 9 \cdot c$, where c is the number of non-zero carries. We now compute $S(a) + S(b) = 1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 = 40$, so the possible values of $S(a + b)$ are 40, 31, 22, 13 and 4.

It is, however, impossible to achieve 40 since 9 added to any non-zero digit is greater than or equal to 10, while 31 is attainable for instance as follows: $9678 + 4321 = 13999$; $1 + 3 + 9 + 9 + 9 = 31$. The result thus reads 31.

Problem 32. In a single-elimination tennis tournament, eight players are randomly allocated to the eight free ends at the bottom of the graph in the picture. Then three rounds are played according to the graph—it is always the winner of a match who continues to the next round. In our tournament there are two professional players and six amateurs, one of whom is Bono. Any professional player always beats an amateur and every two professionals or two amateurs are evenly matched. What is the probability that Bono will play in the final round?



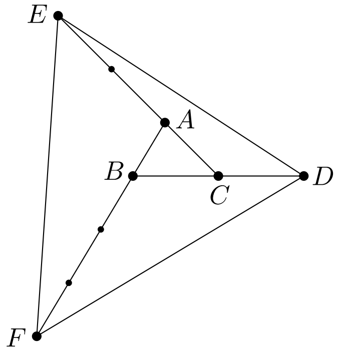
Result. $\frac{1}{14}$
Solution. Let us consider the half of players containing Bono. Bono will play the final if and only if both of the professional players start in the other half and he beats the other three amateurs in his half.

Let us start with only Bono's position fixed and assign the two professional players. The probability that both go to the other half is $\frac{4}{7} \cdot \frac{3}{6}$. Using the fact that all the amateurs are evenly matched, the probability that Bono wins two matches against them is $\frac{1}{2} \cdot \frac{1}{2}$. The desired probability is therefore $\frac{4}{7} \cdot \frac{3}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{14}$.

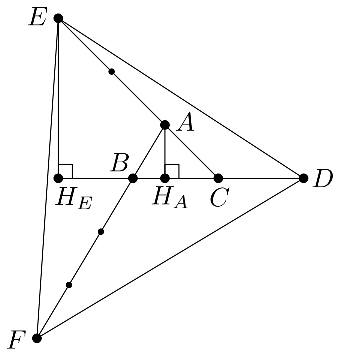
Problem 33. There are numbers
$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$$
written on the blackboard. In each step, we take two of the numbers, say a and b, erase them and write the number
$$\frac{ab}{a + 2ab + b}$$
instead of them. This procedure is repeated until there is only one number left on the blackboard. Find all possible values of that number.

Result. $\frac{1}{5248}$
Solution. Notice that when n replaces a and b, we have
$$\frac{1}{n} = \frac{a + 2ab + b}{ab} = \frac{1}{a} + \frac{1}{b} + 2.$$
It follows that the sum of the reciprocals of all numbers on the blackboard increases by two in each step. As there were 99 steps in total, the last number l satisfies the equation
$$\frac{1}{l} = 2 \cdot 99 + 1 + 2 + \cdots + 100 = 198 + \frac{101 \cdot 100}{2} = 5248.$$
Thus the last number is equal to $\frac{1}{5248}$.

Problem 34. Given a triangle ABC of area 1, extend its sides BC, CA, AB to points D, E, and F respectively, as in the figure, so that $BD = 2BC$, $CE = 3CA$ and $AF = 4AB$. Find the area of the triangle DEF.



Result. 18
Solution. Denote by H_A and H_E the perpendicular projections of A and E, respectively, onto line BC.



The right-angled triangles CAH_A and CEH_E are similar, therefore $CE = 3CA$ implies $EH_E = 3AH_A$. From $CD = BD - BC = BC$ we get that the area of triangle CDE is equal to
$$\tfrac{1}{2} \cdot CD \cdot EH_E = 3 \cdot \tfrac{1}{2} \cdot BC \cdot AH_A = 3.$$
A similar argument then shows that the areas of triangles AEF and BFD are $2 \cdot 4 = 8$ and $3 \cdot 2 = 6$, respectively, so that the total area is $1 + 3 + 8 + 6 = 18$.

Problem 35. The royal tax collectors have collected three bags containing hundreds of golden coins. Each coin in the first, the second and the third bag weighs 10, 11 and 12 grams, respectively. Unfortunately, the labels on the bags got lost. The king has a scale, which shows the weight in grams up to the maximal weight of $N \in \mathbb{N}$ grams; if the weight is bigger than N, the scale simply shows N. The king's intention is to determine which bag contains which type of coins by taking some coins from the bags and a single weighing. What is the minimum value of N such that he can always achieve this?

Result. 47

Solution. Let us first denote the numbers of coins from the bags weighted by the king by a, b, and c. First note that the numbers must be pairwise distinct, for if two of them were equal, the two respective bags would be indistinguishable. We shall call such triplets *admissible*. We search for the smallest possible choice of a, b, and c such that $a \cdot k + b \cdot l + c \cdot m$ are pairwise distinct numbers for any permutation (k, l, m) of the numbers 10, 11 and 12.

The smallest admissible triplet is $a = 0$, $b = 1$ and $c = 2$. This, however, violates the second condition since $32 = 2 \cdot 10 + 12 = 2 \cdot 11 + 10$. The second smallest admissible triplet is $a = 0$, $b = 1$, and $c = 3$, which satisfies the second condition, since

$$\begin{aligned} 3 \cdot 10 + 11 &= 41, & 3 \cdot 10 + 12 &= 42, \\ 3 \cdot 11 + 10 &= 43, & 3 \cdot 11 + 12 &= 45, \\ 3 \cdot 12 + 10 &= 46, & 3 \cdot 12 + 11 &= 47. \end{aligned}$$

The scale thus has to measure correctly up to 47 grams.

Problem 36. Emma has decided to go on a pineapple diet. Every day at 1 pm she checks how many pineapples she has left. If she has at least one, she eats one. If not, she buys one more than she had any day before instead. She bought her first pineapple on day 1 at 1 pm. How many pineapples did she have on day 2020 at 2 pm?

Result. 59

Solution. Let us denote the number of Emma's pineapples on day i at 2 pm by $s(i)$. Since the first two zeros appear on days 2 and 5 and Emma increases every time the number of bought pineapples by one, the distances between two subsequent zeroes in the sequence $s(i)$ form the sequence $3, 4, 5, \ldots$. The zeroes are thus located at positions

$$2 + 3 + 4 + \cdots + n = 1 + 2 + 3 + 4 + \cdots + n - 1 = \frac{n(n+1)}{2} - 1.$$

Since we want to know where the last zero before 2020 appears, we solve the quadratic equation $\frac{1}{2}x(x+1) - 1 = 2020$. It has a negative solution (which is irrelevant) and a positive solution $\frac{\sqrt{16169}}{2} - \frac{1}{2}$ lying between 63 and 64. Thus the last zero is the 62nd and it is at the position $\frac{63 \cdot 64}{2} - 1 = 2015$. The sequence $s(i)$ then continues as follows: $63, 62, 61, 60, 59, \ldots$, so the sought number $s(2020)$ equals 59.

Problem 37. Pig farmer Joe has a new pigsty of area $252\,\mathrm{m}^2$ for his young pigs. Inside his pigsty he has flexibly movable dividing walls such that there are 16 rectangular boxes. These dividing walls can only be moved parallel to the outer walls along the entire length and width of the pigsty. Now he has moved the dividing walls so that some of the boxes have sizes in m^2 as in the picture. As an amateur mathematician, he always takes care that the pig boxes have sides of positive integer sizes. Find all possible areas of the pig box in the upper right corner containing the question mark (in m^2).

24			?
18		12	
			12
30	10		

Result. 8, 24

Solution. First note that we know the ratio of width of the first and second columns (30 : 10) and of the first and third columns (18 : 12). We also know the ratio of heights of the first, second and fourth rows (24 : 18 : 30). Moreover, the reduction of the ratios to the lowest terms produces 3 : 1 : 2 and 4 : 3 : 5 for the columns and rows, respectively. Knowing that, we can fill in the table with areas of the corresponding cells as in the figure, where the third row and fourth column are specified only partially with integer parameters x and y to reflect the ratios.

24	8	16	$4y$
18	6	12	$3y$
$3x$	x	$2x$	12
30	10	20	$5y$

The ratio of the height of the second and third row could be expressed comparing the areas in the third column as $12:2x$ or from the fourth columns as $3y:12$, see the shaded rectangle above. These two values must, of course, be the same, so we write $12:2x = 3y:12$, i.e. $y = 24/x$.

Finally, we compare the total area of the pigsty with the sum of all the boxes to get the equation

$$96 = 6x + 12y = 6x + 12\frac{24}{x},$$

which could be rewritten to

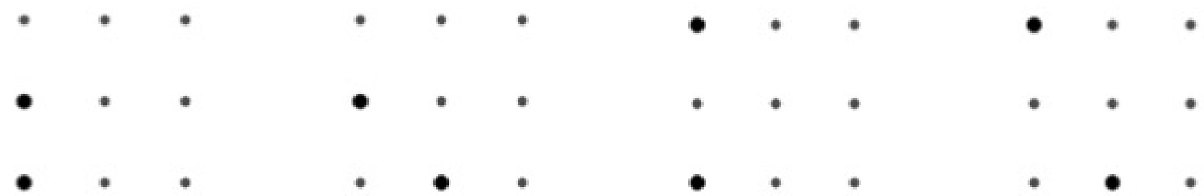
$$0 = x^2 - 16x + 48 = (x-4)(x-12).$$

The solution is thus $x = 4$ or $x = 12$, which lead to $y = 6$, $? = 24$ and $y = 2$, $? = 8$, respectively.

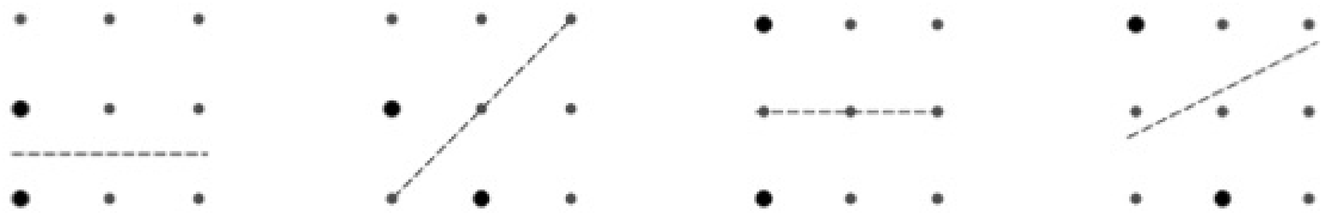
Problem 38. Daniel and Philip both drew a circle on a piece of paper with a grid made by 1×1 squares. Both circles pass through exactly three grid points. Daniel's circle has radius $\frac{5}{4}$, Philip's circle is even smaller. What is the radius of Philip's circle?

Result. $\frac{5\sqrt{2}}{6}$

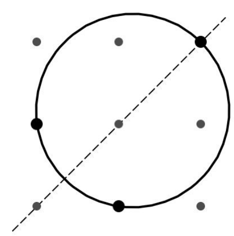
Solution. Let A, B, C denote the grid points on Philip's circle. We are given that the radius is less than $\frac{5}{4}$, so the distance between A and B is at most $\frac{5}{2}$, and therefore, up to a rotation, the relative position of A and B is one of the four arrangements shown below:



In each case, we can find a line of reflectional symmetry for the circle, and the third grid point C must be placed either on this line, or in such a way that its reflection across the line is not a fourth grid point. This makes arrangement 1 impossible.



Arrangement 2 is possible, as long as we place C on the line of symmetry. Moreover, we must also position A, C and B, C in one of the arrangements 2 through 4, up to rotation. This leads us to Philip's circle:



We still need to find the radius, which we can do algebraically. We introduce a coordinate system so that the points A, B, C have coordinates $(1, 0)$, $(0, 1)$, and $(2, 2)$. Substituting these (x, y)-values into the general equation of a circle $(x-h)^2 + (y-k)^2 = r^2$, we solve for h, k, and r, and arrive at the equation $(x - \frac{7}{6})^2 + (y - \frac{7}{6})^2 = \frac{25}{18}$.

Problem 39. Seven imps wear hats of seven distinct colours. A malicious wizard Colorius wants to devise a spell changing the colours of the hats so that

- the new colour of each hat depends only on its previous colour, not on who wears it or what the other hats look like,
- after the spell takes effect, all seven original colours are still present, and
- when Colorius casts the same spell two or three times in a row, in neither case will any of the imps wear a hat of his original colour.

How many such spells can Colorius devise?

Result. 720

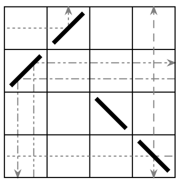
Solution. Since all colours remain present after the spell, we infer that the spell is just a permutation of the seven original colours. Every such permutation can be decomposed into disjoint oriented cycles of colours such that the permutation simply rotates these cycles. It follows from the third condition in the statement that these cycles cannot be formed by 2 or 3 colours, but a 1-colour cycle is impossible, too. However, it is impossible to distribute the seven colours into more than one cycle so that all the cycles contain at least four colours, so the permutation in question is formed by a single cycle. Finally, there are $6! = 720$ such permutations: Fix one of the colours, there are six options to which colour the spell can change it, for this colour there are five options etc., until the last colour gets changed to the first one.

Problem 40. Mary has a combination lock—but it is no ordinary lock, as each of its rings has a different number of numbers on it. The first ring has numbers from 0 to 4, the second one from 0 to 6, the third one from 0 to 10, and the fourth one from 0 to 9. Mary knows that if she sets the lock rings to show $0, 0, 0, 0$ and starts rotating all rings simultaneously (so that the next combination shown is $1, 1, 1, 1$), she will eventually get to a combination which ends with $5, 1$. She also knows that when this happens for the second time, the rings show the combination to unlock the lock. Help her and find the unlocking combination.

Result. $1, 6, 5, 1$

Solution. We are looking for an integer x such that x gives remainders 5 and 1 when divided by 11 and 10, respectively. Since 10 and 11 are coprime, by the Chinese remainder theorem, there is precisely one such x among the numbers $0, 1, \ldots, 109$ (let us call it x_0) and all the other solutions are obtained by adding multiples of $10 \cdot 11 = 110$ to x_0. A possible easy way to find x_0 is to list integers of the form $11k + 5$ (k an integer) and pick the one ending with 1, which turns out to be 71. This is the number of rotations needed to be done to get the first combination ending with $5, 1$. By the above, such a combination will come up again after 110 rotations (and not earlier), i.e. $110 + 71 = 181$ rotations from the beginning. Since 181 gives remainders 1 and 6 when divided by 5 and 7, respectively, at that point the lock will show $1, 6, 5, 1$.

Problem 41. In some squares of a 4×4 table four double-sided mirrors have been placed diagonally. From each of the sixteen segments on the boundary of the table, a ray of light has been released perpendicularly to the segment. The ray goes straight and changes its direction by 90° every time it hits a mirror. It occurred that exactly four of these rays had one end on the bottom side of the table and the other on the right side, another four had ends on the right and the upper side, another four ended on the upper and the left side and finally, four had one end on the left side and one on the bottom side. For how many different configurations of the four mirrors does this happen? (The picture shows one such configuration with some of the rays.)



Result. 144

Solution. Since there are only four mirrors and each ray has to turn at some point, every row and every column has to contain exactly one mirror. This can be achieved in $4! = 24$ different ways—we choose the column for the first row in 4 ways, then the column for the second in 3 ways and so on. Due to the restrictions on the number of rays going in each direction, we know that there must be two mirrors of each type. Hence having chosen the squares to which the mirrors are to be placed, we have $\binom{4}{2} = 6$ ways to choose the orientation. Finally, it is not hard to see that every such configuration has the desired properties, hence there are $24 \cdot 6 = 144$ such configurations.

Problem 42. Let $x_1 = 2020$ and let x_n be equal to x_{n-1} multiplied by the smallest prime p, which is not a divisor of x_{n-1}, and divided by all primes smaller than p. Find the number of different prime divisors of x_{2020}.

Result. 9

Solution. We see that $x_1 = 2 \cdot 2 \cdot 5 \cdot 101$ and $x_2 = 2 \cdot 3 \cdot 5 \cdot 101$. It is easy to see that when a term of the sequence is not divisible by a square of a prime number, all the subsequent terms have this property, too. Therefore, from x_2 on we can represent every term x_n as a binary number b_n, whose k-th digit (from the right) is 1 if and only if x_n is divisible by the k-th prime number. Next, observe that according to the definition of the sequence, $b_{n+1} = b_n + 1$ for all $n \geq 2$. We have

$$b_2 = 10000000000000000000000111_2$$

and

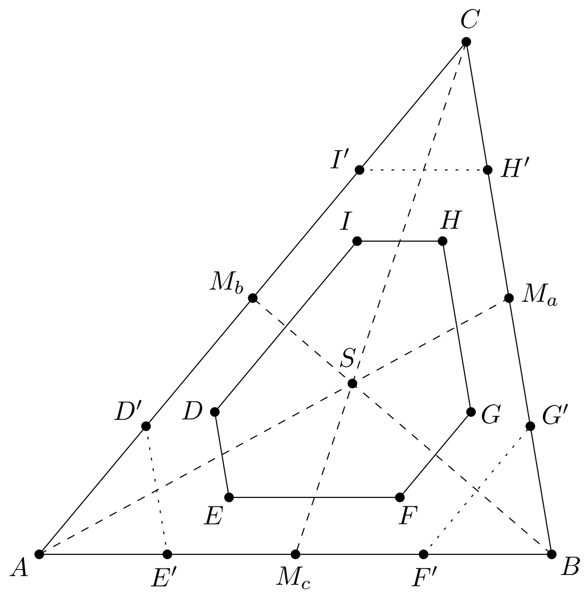
$$b_{2020} = b_2 + \underbrace{11111100010_2}_{=2018} = 10000000000000011111101001_2.$$

From the definition of b_n, the number of different prime divisors of x_{2020} is just the number of ones in b_{2020}, which is 9.

Problem 43. The medians of triangle ABC dissect it into six sub-triangles. The centroids of these sub-triangles are vertices of hexagon $DEFGHI$. Find the area ratio between the hexagon $DEFGHI$ and the triangle ABC.

Result. $\frac{13}{36}$

Solution. The following figure represents the situation containing all the relevant points: the centroid S of triangle $\triangle ABC$, the midpoints M_a, M_b, M_c of the sides of $\triangle ABC$, the centroids D, E, F, G, H, I of the sub-triangles, the midpoints $D', E', \ldots, I'$ of the line segments $M_bA, AM_c, \ldots, CM_b$, respectively.



Since $AE' = \frac{1}{4}AB$ and $AD' = \frac{1}{4}AC$, triangle $\triangle AE'D'$ is a homothetic transformation of $\triangle ABC$ with center A and scaling factor $\frac{1}{4}$. Therefore, the area $[AE'D']$ is $\frac{1}{16}[ABC]$. Likewise, we have $[BG'F'] = [CI'H'] = \frac{1}{16}[ABC]$. Therefore, the area of hexagon $D'E'F'G'H'I'$ is $\frac{13}{16}[ABC]$. Furthermore, the homothetic transformation with center S and scaling factor $\frac{3}{2}$ maps hexagon $DEFGHI$ to hexagon $D'E'F'G'H'I'$ and we obtain the area of hexagon $DEFGHI$ as

$$\frac{4}{9} \cdot \frac{13}{16}[ABC] = \frac{13}{36}[ABC].$$

As a consequence, the area ratio sought is $\frac{13}{36}$.

Problem 44. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers such that $a_{m+1} = m(-1)^{m+1} - 2a_m$ for all positive integers m and $a_1 = a_{2020}$. Find the value of $a_1 + a_2 + \cdots + a_{2019}$.

Result. $\frac{1010}{3}$

Solution. Adding up the equations for $m = 1, \ldots, 2019$ we obtain

$$(a_2 + a_3 + \cdots + a_{2020}) = (1 - 2 + 3 - \cdots + 2019) - 2(a_1 + a_2 + \cdots + a_{2019}).$$

Using $a_1 = a_{2020}$, we can rearrange this as follows:

$$3(a_1 + a_2 + \cdots + a_{2019}) = 1 - 2 + 3 - \cdots + 2019 = (1-2) + (3-4) + \cdots + (2017-2018) + 2019 = (-1) \cdot 1009 + 2019 = 1010.$$

Thus

$$a_1 + a_2 + \cdots + a_{2019} = 1010/3.$$

Note that such a sequence of real numbers indeed exists: Given a_1, the rest of the sequence is determined by the equation in the statement. One can therefore express a_{2020} in terms of a_1 only and the condition $a_1 = a_{2020}$ then translates to a linear equation in a_1; it is not difficult to see that this equation has a solution.

Problem 45. Sandra holds five identical strings in her hand so that each string has one end on each side of her hand. She asks Will to tie pairs of ends on either side at random until only one end on each side is remaining. At most two ends of strings can be tied together. How likely are the strings to fit together in a single long thread?

Result. 8/15

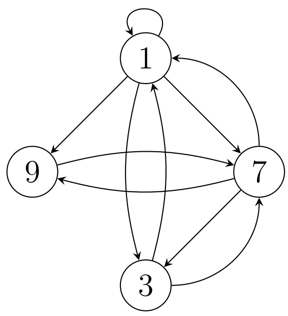
Solution. Assume that the strings, which we label A, B, C, D, and E are tied on one side of the hand so that A has a free end, B is tied to C and D is tied to E. Observe that under such circumstances, we obtain one thread if and only if (on the other side) we tie A to one of B, C, D, E (4 options) and subsequently tie the remaining free end of the pair to a string of the other pair (2 options)—for example, if A is tied to B, then we tie C with D or E. Thus there are $4 \cdot 2 = 8$ options.

The total number of ways to tie the knots on the other side is 15: First we choose the free end (5 options) and the rest is determined by pairing one particular end with one of the three remaining ends (3 options). We conclude that the probability of forming a single thread is 8/15.

Problem 46. Call a number a *2-prime* if any pair of its consecutive digits forms a different two digit prime number. For example, 237 is 2-prime, while 136 and 1313 are not. Find the largest 2-prime number.

Result. 619737131179

Solution. Let us consider an oriented graph on 4 vertices labelled 1, 3, 7 and 9 where two digits are connected with an arrow if and only if the associated two-digit number is a prime. Note that there is a loop at vertex 1.



Assume for the moment that there is a walk on the graph moving along the arrows and using every arrow exactly once. Then the largest 2-prime can be obtained by putting 6 or 8 in front of the sequence of digits visited by one of such walks. Indeed, it follows from the definition of 2-prime that all its digits except for the first one must be taken from the set $\{1, 3, 7, 9\}$ and no arrow can be used twice. Thus no 2-prime can have more digits than the number of arrows in our graph plus two (one for the first digit and one because we are counting digits), i.e. 12. Also, the first digit cannot be 9 or 7 (it would repeat one of the arrows) and since 61, 83, 87 and 89 are prime numbers, we do not need smaller digits.

Now we find the walk with the above-mentioned properties which produces the largest possible number. Note that on one hand, there is one more arrow entering vertex 9 than leaving it while, on the other hand, one more arrow leaving vertex 1 than entering it. The other two vertices are "balanced" in this sense. It follows that our walk must start in 1 and end in 9. From 1 we move to 9 as it is the largest possible neighbour, then we continue to 7 for the same reason, then we cannot return to 9 (it would terminate the sequence) so we rather move to 3, then back to 7 etc. By this sort of greedy algorithm we end up with 19737131179 and as 81 is not prime, we conclude that the largest 2-prime is 619737131179.

Problem 47. Let O be the circumcenter of triangle ABC. Let further points D and E lie on the segments AB and AC, respectively, so that O is the midpoint of DE. If $AD = 8$, $BD = 3$, and $AO = 7$, determine the length of CE.

Result. $\frac{4\sqrt{21}}{7}$

Solution. Consider a reflection with respect to the circumcenter O and denote the respective images of points by adding a prime. We observe that points A' and B' lie on the circumcircle of $\triangle ABC$ and $D' = E$ (i.e. $E' = D$). The Pythagorean theorem in the right-angled triangle $AA'B'$ (note that AA' is a diameter of the circumcircle) yields

$$(AB')^2 = (AA')^2 - (A'B')^2 = 14^2 - AB^2 = 14^2 - 11^2 = 75.$$

Since $\angle AB'E = \angle A'BD = 90°$, the Pythagorean theorem in $\triangle AB'E$ yields

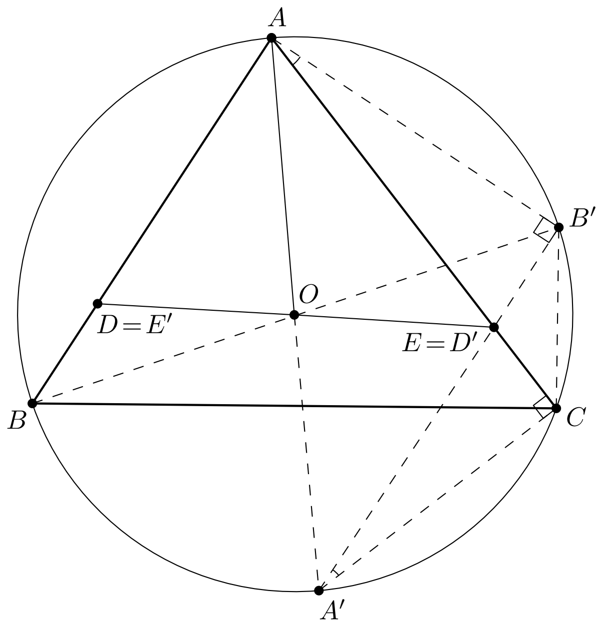
$$AE = \sqrt{(AB')^2 + BD^2} = \sqrt{75+9} = 2\sqrt{21}.$$

Since E (the image of D) lies on $A'B'$ (the image of AB) and since the quadrilateral $AB'CA'$ is cyclic, it follows that $\triangle AB'E \sim \triangle A'CE$. Hence

$$\frac{CE}{A'E} = \frac{B'E}{AE}$$

and we conclude that

$$CE = AD \cdot \frac{BD}{AE} = 8 \cdot \frac{3}{2\sqrt{21}} = \frac{4\sqrt{21}}{7}.$$



Alternative solution: The power of point D with respect to the circumcircle of $\triangle ABC$ with radius $r = AO$ equals

$$-3 \cdot 8 = -DB \cdot DA = OD^2 - r^2 \Rightarrow OE = OD = \sqrt{49-24} = 5$$

(the minus sign is present due to the fact that D lies inside the circle). The Law of Cosines in $\triangle ADO$ yields

$$8^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7\cos(\angle AOD).$$

Since $\cos(\angle AOE) = \cos(180° - \angle AOD) = -\cos(\angle AOD) = -\frac{1}{7}$, the same theorem for $\triangle AOE$ then yields

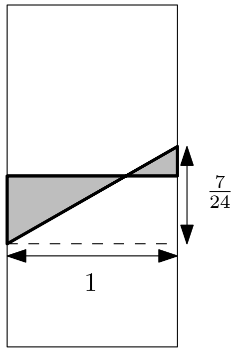
$$AE^2 = 7^2 + 5^2 - 2 \cdot 5 \cdot 7\cos(\angle AOE) = 84 \Rightarrow AE = \sqrt{84} = 2\sqrt{21}.$$

Similarly as above, from power of point E with respect to the circumcircle of $\triangle ABC$ we obtain

$$-2\sqrt{21} \cdot EC = 5^2 - 7^2 \Rightarrow EC = \frac{24}{2\sqrt{21}} = \frac{4\sqrt{21}}{7}.$$

Problem 48. Rectangle of dimensions 7×24 is divided into squares 1×1. One of its diagonals cuts triangles from some of the squares. Find the sum of perimeters of all these triangles.
Result. $\frac{56}{3} = 18\frac{2}{3}$
Solution. Let 24 be the width and 7 the height of the rectangle. The diagonal going from the lower left to the upper right corner has a slope of $\frac{7}{24}$. When it passes through a square, it cuts off a triangle if and only if it crosses a horizontal side that separates two squares. Since the slope is constant, we can rearrange the line segments of the two triangles that are cut off from the two squares into one big right triangle of width 1. Then the legs are 1 and $\frac{7}{24}$ and the hypotenuse

is $\sqrt{1+\left(\frac{7}{24}\right)^2} = \frac{25}{24}$ long.



The sum of the perimeters of these two triangles is thus $\frac{56}{24}$. The diagonal crosses a horizontal side exactly six times, plus two big right triangles of the above dimensions are cut off from the first and last squares that the diagonal passes. So the total sum of the perimeters is $8 \cdot \frac{56}{24} = \frac{56}{3}$.

Problem 49. Let us call a positive integer n *elevating* if it is possible to get from each floor of an 8787-storey building to any other when it is only allowed to go 2020 floors down or n floors up. Find the largest elevating number.

Note: A k-storey building has k floors above the ground level and a ground floor.

Result. 6763

Solution. Firstly, in order to be able to move at all from floor 2019, we must have $2019 + n \le 8787 \Rightarrow n \le 6768$. Secondly, the condition $d := \gcd(2020, n) = 1$ is necessary, as we can move between floors a and b only if $d \mid a - b$. Bearing in mind that $2020 = 2^2 \cdot 5 \cdot 101$, we can eliminate some candidates for the largest n: 6768 divisible by 2, 6767 divisible by 101, 6766 divisible by 2, 6765 divisible by 5, 6764 divisible by 2 and finally $\gcd(6763, 2020) = 1$.

It remains to prove that 6763 is elevating. Using the Euclidean algorithm (or Bézout's identity) we can find integers x, y such that $6763x - 2020y = 1$ and we can assume that x, y are even non-negative since adding 2020 to x and 6763 to y preserves the equality. Starting in floor $0 \le f \le 8786$, we claim that is possible to make a sequence of x moves up and y moves down in such order that we stay in the building and end in floor $f + 1$. Indeed, as $2020 + 6763 \le 8787$, we can always move in at least one direction. Furthermore, if we used all of the steps down (resp. up), we must be below (resp. above) floor $f + 1$ and making the remaining steps up (resp. down) brings us to floor $f + 1$. Similarly it can be shown that we can move one floor down from any floor $1 \le f \le 8787$. We conclude that 6763 is the largest elevating number.

Problem 50. In the cryptogram

$$\begin{array}{cccccc} & & & R & E & D \\ + & & B & L & U & E \\ + & G & R & E & E & N \\ \hline = & B & R & O & W & N \end{array}$$

different letters represent different digits. None of the four numbers may start with zero. Furthermore, we know that $BLUE$ is a perfect square. Find the five-digit number $BROWN$.

Result. 85230

Solution. We number the columns from left to right by 1 up to 5 and denote the carries of the respective columns by $c_1, \dots, c_5$. Observe that $c_1 = 0$ and $0 \le c_i \le 2$ for $i = 2, \dots, 5$: Indeed, one cannot get more than 29 by adding three digits and a carry of size at most 2 and thus the last inequality follows from an inductive argument. Further note that $c_2 \le 1$—as there are two different digits summed up in the second column and $c_3 \le 2$—and $c_2 \ne 0$ due to the first column, so $c_2 = 1$ and $G + 1 = B$. From the fifth column we get $D + E = 10$, since D and E cannot be both zeroes, and $c_5 = 1$. Since $B + R + c_3 = c_2 \cdot 10 + R$, either $B = 9$ or $B = 8$. Therefore $BLUE$ is the square of an integer n satisfying $90 \le n \le 99$ and has four different digits. Eliminating the squares with coincident digits, we are left with 8649, 9025, 9216, 9604, and 9801 as possible values for $BLUE$. From $D + E = 10$ we can eliminate 9025, since this would lead to $D = E\ (= 5)$. Furthermore, 9801 is impossible due to $B = D\ (= 9)$ and 9604 due to $L = D\ (= 6)$. With the help from the fourth column we can eliminate 9216, because this would lead to $D = 4$ and $E + U + E + 1 = 6 + 1 + 6 + 1 = 14$ giving $W = 4$, which is a contradiction with $D = 4$. Therefore the only possible value of $BLUE$ is 8649.

From $B = 8$, $L = 6$, $U = 4$, $E = 9$ we easily get $D = 1$, $W = 3$, and $G = 7$ and the carries $c_3 = c_4 = 2$. The third column now gives $R + L + E + c_4 = c_3 \cdot 10 + O$, which can be simplified to $R + 17 = 20 + O$ and this is possible with $R = 5$ and $O = 2$ only. As a consequence, we finally get $N = 0$ and the cryptogram has the unique solution

$$\begin{array}{cccccc} & & & 5 & 9 & 1 \\ + & & 8 & 6 & 4 & 9 \\ + & 7 & 5 & 9 & 9 & 0 \\ \hline = & 8 & 5 & 2 & 3 & 0 \end{array}$$

Problem 51. Find the smallest positive integer $k > 1$ such that there is no positive k-digit integer n with every digit odd and $S(S(n)) = 2$, where $S(x)$ denotes the sum of digits of x.

Result. 103

Solution. Firstly observe that $S(m) = 2$ for an odd integer m if and only if $m = 10^l + 1$ for some positive integer l. If $k = 103$, then $S(n)$ is necessarily odd for any k-digit n with all digits odd, hence for $S(S(n)) = 2$ to hold, $S(n)$ has to be of the form above. However,

$$101 < 103 \cdot 1 \leq S(n) \leq 103 \cdot 9 = 927 < 1001$$

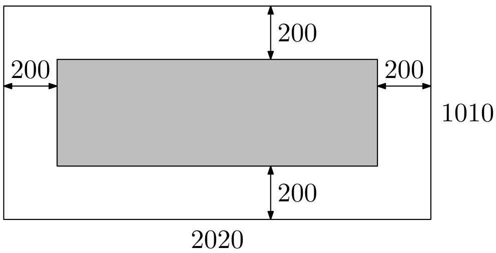
for every n with 103 digits. Therefore $k = 103$ satisfies the condition from the statement.

We will now prove that for odd $k < 103$, there exists n as described in the statement. It is easy to see that $S(n)$ can attain any odd value greater or equal to k and less or equal to $9k$. If $1 < k \leq 11$, then $9k \geq 18 > 11$, so $S(n)$ can be equal to 11 and consequently, there exists n such that $S(S(n)) = 2$. If $101 \geq k > 11$, then $9k \geq 9 \cdot 13 = 117 > 101$, so $S(n)$ can be equal to 101 and again $S(S(n)) = 2$. So $k > 101$.

If $k < 103$ is even, the reasoning is basically the same, the only difference is that $S(n)$ is even. For $k = 2$ we can use $n = 11$. If $2 < k \leq 20$, then $9k > 20$, so we can find n with $S(n)$ equal to 20. If $103 > k > 20$, then $9k > 180 > 110$, so $S(n)$ can be equal to 110.

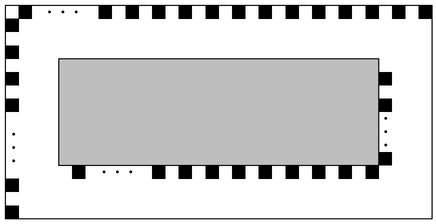
This shows that 103 is the sought smallest number having the desired property.

Problem 52. Martin bought a chessboard, which was formed by a rectangle consisting of 1010×2020 squares, out of which a smaller rectangle had been removed as in the figure below. He placed a bug on every square of the chessboard. However, some of the bugs had a cough. To make things worse, the cough was very infectious: Every bug sitting on a square neighbouring at least two squares with coughy bugs got the cough as well. (We say that two squares are neighbouring if they share a side.) Determine the least possible number of bugs that could infect all others. The bugs did not move.



Result. 2630

Solution. Observe that when a bug is infected by the described procedure, the total perimeter of the "contaminated" region does not increase. Therefore, there had to be at least $P/4$ infectious bugs initially, where P is the perimeter of the "O" shape. We can easily compute that $P = 2(2020 + 1010 + (2020 - 400) + (1010 - 400)) = 10520$ and the picture shows an arrangement of $P/4 = 2630$ coughy bugs that would infect all others.



Problem 53. A positive integer has 25! distinct positive divisors. Find at most how many of them may be the 5th powers of a prime number.

Note: The symbol $n!$ denotes the product of all positive integers less than or equal to n.

Result. 27

Solution. A number $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where p_i are distinct primes has $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ positive divisors. So we see that the maximal number of 5th powers of a prime dividing our number is equal to the maximal number of factors ≥ 6 in some factorisation of 25!. In order to maximise this number, we consider the prime factorisation

$$25! = 2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23,$$

leave primes larger then 5 and combine 5's and 3's each with a 2 and finally write 2^6 as 8^2 to get the maximal number 27.

Problem 54. Positive real numbers x, y, z satisfy

$$x^2 + xy + y^2 = 1,$$
$$y^2 + yz + z^2 = 2,$$
$$z^2 + zx + x^2 = 3.$$

Find the value of $xy + yz + zx$.

Result. $2\sqrt{2/3} = \frac{2}{3}\sqrt{6}$

Solution. By adding the first and the third equation and subtracting twice the second one, we obtain

$$(2x - y - z)(x + y + z) = 0$$

and since x, y, z are positive, $2x = y + z$. Put $y = x - \delta$, $z = x + \delta$ and plug in to the equations from the statement to obtain

$$3x^2 - 3x\delta + \delta^2 = 1,$$
$$3x^2 + \delta^2 = 2,$$
$$3x^2 + 3x\delta + \delta^2 = 3.$$

It follows that $x\delta = 1/3$ and plugging into the second new equation gives a quadratic equation with solutions

$$\delta^2 = 1 \pm \tfrac{1}{3}\sqrt{6}.$$

We are asked to compute $3x^2 - \delta^2 = 2 - 2\delta^2$ and since the result has to be positive, the only possibility is $2\sqrt{2/3}$.

Alternative solution: Pick a point P in the plane and draw line segments PA, PB and PC of lengths x, y and z respectively, so that $\angle APB = \angle BPC = \angle CPA = 120°$. Since $\cos(120°) = -\frac{1}{2}$, the given equations and the Law of Cosines yield $AB = 1$, $BC = \sqrt{2}$ and $AC = \sqrt{3}$ and thus ABC is a right triangle with area $S = \frac{\sqrt{2}}{2}$. Note that this area can be computed also using the three triangles sharing vertex P as follows: $S = \frac{1}{2}\sin(120°)(xy + yz + zx)$. Recall that $\sin(120°) = \frac{\sqrt{3}}{2}$ and conclude that $xy + yz + zx = \frac{2\sqrt{6}}{3}$.

Problem 55. Let I and O be respectively the incenter and the circumcenter of triangle ABC with the properties $AB = 495$, $AC = 977$ and $\angle AIO = 90°$. Determine the length of BC.

Result. 736

Solution. Consider a homothety with center A and ratio 2 and mark respective images of points by adding a prime. Then AO' is clearly the diameter of the circumcircle of $\triangle ABC$ and as $\angle AIO = 90°$, I' also lies on the circumcircle. It follows that it is the midpoint of arc BC not containing A. It is well known that this point (from now on denoted by $\check{S}$) satisfies $\check{S}I = \check{S}C$ and basic angle chasing reveals $\angle BC\check{S} = \angle BA\check{S} = \angle CA\check{S}$. Let D denote the intersection of $A\check{S}$ and BC. Then $\triangle D\check{S}C \sim \triangle C\check{S}A$ and hence

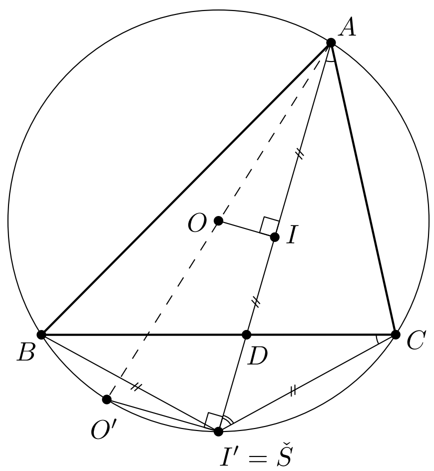
$$\frac{\check{S}D}{\check{S}I} = \frac{D\check{S}}{\check{S}C} = \frac{C\check{S}}{\check{S}A} = \frac{\check{S}I}{\check{S}A} = \frac{1}{2}$$

is the inverted ratio of the homothety. It follows that $[BCI] = \frac{1}{3}[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ. This can be rewritten using the radius r of the incircle of $\triangle ABC$ as

$$\frac{1}{2}r \cdot BC = \frac{1}{6}r(AB + BC + CA)$$

and so

$$BC = \frac{AB + AC}{2} = \frac{495 + 977}{2} = 736.$$



Problem 56. Find all triples (a, b, c) of positive integers satisfying the equation $3abc = 2a + 5b + 7c$.

Result. $(1, 16, 2)$, $(2, 11, 1)$, $(12, 1, 1)$

Solution. Dividing the equation by the (positive) number abc yields

$$3 = \frac{2}{bc} + \frac{5}{ca} + \frac{7}{ab}.$$

If all three unknowns are larger than one and at least one of them is larger than two, then the right-hand side is at most

$$\frac{2}{2 \cdot 3} + \frac{5}{2 \cdot 3} + \frac{7}{2 \cdot 2} = \frac{35}{12} < 3,$$

so there is no solution under these assumptions. One can easily verify that $a = b = c = 2$ does not yield a solution either. Therefore at least one of a, b, c is equal to 1.

If $a = 1$, then the original equation reads

$$3bc = 2 + 5b + 7c,$$

which, after multiplying by 3 and rearranging, can be factorised to

$$(3b - 7)(3c - 5) = 41.$$

Since both factors have to be positive divisors of prime number 41, which gives remainder 2 when divided by 3, we have only one solution, namely $b = 16$ and $c = 2$.

If $b = 1$, we obtain

$$3ac = 2a + 5 + 7c$$

and a similar step yields

$$(3a - 7)(3c - 2) = 29,$$

leading to $a = 12$ and $c = 1$ using the same divisibility argument as before.

Finally, the option $c = 1$ produces the equation

$$(3a - 5)(3b - 2) = 31$$

with solutions $a = 12$, $b = 1$ and $a = 2$, $b = 11$, the former being already found in the previous step.

Summing up, there are exactly three solutions: $(1, 16, 2)$, $(2, 11, 1)$ and $(12, 1, 1)$.

Problem 57. At a party, every guest is a friend of exactly fourteen other guests (not including him or her). Every two friends have exactly six other attending friends in common, whereas every pair of non-friends has only two friends in common. How many guests are at the party?

Result. 64

Solution. Pick a guest x together with all his or her friends and call this group of 15 people H. Let y be a member of H different from x, we claim that y has precisely 7 friends outside of H: Out of 14 friends of y, one is x and further six are common friends of x and y, all included in H. Therefore there are altogether $c = 14 \cdot 7 = 98$ pairs (y, z), where y is a member of H different from x and z is a friend of y outside H. However, the number c can also be computed as follows: Each guest z outside H has precisely two friends in H, because x is not a friend of z by the definition of H and both the common friends of x and z are in H. In other words, c is twice the number of guests outside H, therefore there are $98/2 = 49$ guests not in H. Since H has 15 members, we conclude that the party is attended by $15 + 49 = 64$ people.

Let us note that such a configuration of relations between 64 guests is indeed possible: Put the guests into the cells of an 8×8 table and let every two guests be friends of each other precisely if they are in the same row or column. It is easy to see that the conditions from the statement are then satisfied.

Problem 58. A point P is located in the interior of triangle ABC. If

$$AP = \sqrt{3}, \quad BP = 5, \quad CP = 2, \quad AB : AC = 2 : 1, \quad \text{and} \quad \angle BAC = 60^\circ,$$

what is the area of triangle ABC?

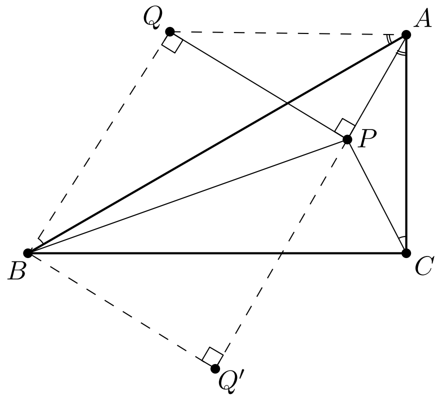
Result. $\frac{6+7\sqrt{3}}{2}$

Solution. Take a point Q on the opposite side from the point C with respect to line AB such that $\triangle ABQ \sim \triangle ACP$. The similarity ratio equals $AB : AC = 2$ and it follows that $AQ = 2AP = 2\sqrt{3}$ and $BQ = 2CP = 4$. Note that these equalities together with $\angle QAB = \angle PAC$ imply $\triangle APQ \sim \triangle ACB$ and hence $\angle APQ = 90°$ and

$$PQ = \sqrt{(2\sqrt{3})^2 - (\sqrt{3})^2} = 3$$

due to the Pythagorean theorem (we will refer to it as the Theorem in the rest of this solution). Since $BP^2 = 5^2 = 4^2 + 3^2 = BQ^2 + PQ^2$, the reverse implication of the Theorem yields $\angle BQP = 90°$. Considering the reflection Q' of Q with respect to the midpoint of BP then allows us to use the Theorem again in right triangle $AQ'B$ to compute $AB^2 = PQ^2 + (AP + BQ)^2 = 28 + 8\sqrt{3}$, and conclude that the area of $\triangle ABC$ equals

$$\frac{1}{2} \cdot AB \cdot AC \cdot \sin 60° = \frac{\sqrt{3}}{8} AB^2 = \frac{6 + 7\sqrt{3}}{2}.$$



Problem 59. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with non-negative integer coefficients such that

$$P\left(\frac{\sqrt{21} - 1}{2}\right) = 2020.$$

Find the minimal possible value of $a_n + a_{n-1} + \cdots + a_1 + a_0$.

Result. 22

Solution. Let $u = \frac{\sqrt{21}-1}{2}$ and note that we are minimizing $a_n + a_{n-1} + \cdots + a_1 + a_0 = P(1)$. We split the solution into several steps.

Step 1: Check that u is a root of $G(x) = x^2 + x - 5$ and divide $P(x)$ by $G(x)$, i.e. write

$$P(x) = Q(x)G(x) + Ax + B$$

for some integers A, B and a polynomial Q with integer coefficients (since the leading coefficient of $G(x)$ is 1, standard algorithm of polynomial long division yields the result).

Step 2: Since $P(u) - 2020 = 0$, A, B are integers and u is irrational, we conclude that $A = 0$ and $B - 2020 = 0$, i.e.

$$P(x) = Q(x)G(x) + 2020. \tag{$\star$}$$

Step 3: If any of the coefficients of $P(x)$, say a_k, satisfies $a_k \geq 5$, then the polynomial $\tilde{P}(x) = P(x) + G(x)x^k = P(x) + (x^2 + x - 5)x^k$ also satisfies all the conditions and $\tilde{P}(1) = P(1) - 3$. Repeating this procedure as many times as possible we end up with a polynomial $P(x)$ with all coefficients satisfying $a_k \in \{0, 1, 2, 3, 4\}$ and $P(u) = 2020$.

Step 4: Observe that the polynomial P satisfying these properties is unique. Indeed, any such polynomial fulfills the equation ($\star$) with an appropriate polynomial $Q(x)$, in order to have $0 \leq a_0 \leq 4$, where a_0 is the constant coefficient of $P(x)$, we infer that the constant coefficient of $Q(x)$ must satisfy $q_0 = 404$. Due to the fact that we know all coefficients of $G(x)$ and the constant one has absolute value 5, we can proceed to determine the linear coefficient q_1 from equation ($\star$) etc. Uniqueness of $Q(x)$ clearly implies the desired uniqueness of $P(x)$.

Step 5: Now it only remains to provide the computations arising from repeating the procedure described in Step 3. We start with constant polynomial $P_0(x) = 2020$ and proceed as follows:

$$
\begin{aligned}
P_1(x) &= 404x^2 + 404x \\
P_2(x) &= 80x^3 + 484x^2 + 4x \\
P_3(x) &= 96x^4 + 176x^3 + 4x^2 + 4x \\
P_4(x) &= 35x^5 + 131x^4 + x^3 + 4x^2 + 4x \\
P_5(x) &= 26x^6 + 61x^5 + x^4 + x^3 + 4x^2 + 4x \\
P_6(x) &= 12x^7 + 38x^6 + x^5 + x^4 + x^3 + 4x^2 + 4x \\
P_7(x) &= 7x^8 + 19x^7 + 3x^6 + x^5 + x^4 + x^3 + 4x^2 + 4x \\
P_8(x) &= 3x^9 + 10x^8 + 4x^7 + 3x^6 + x^5 + x^4 + x^3 + 4x^2 + 4x \\
P_9(x) &= 2x^{10} + 5x^9 + 4x^7 + 3x^6 + x^5 + x^4 + x^3 + 4x^2 + 4x \\
P_{10}(x) &= x^{11} + 3x^{10} + 4x^7 + 3x^6 + x^5 + x^4 + x^3 + 4x^2 + 4x.
\end{aligned}
$$

The sought minimum is thus $P_{10}(1) = 1 + 3 + 4 + 3 + 1 + 1 + 1 + 4 + 4 = 22$.