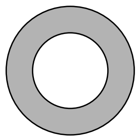
Problem 1. If we increase the length of each edge of a cube by 100%, by what percent does the volume increase?

Result. 700%

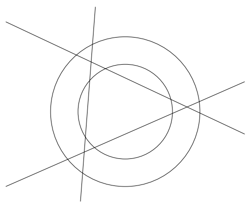
Solution outline. Increasing the length of an edge a by 100% is the same as multiplying it by 2. The volume of the cube is now a^3, after multiplication it will be $(2a)^3 = 8a^3$. There is an increase by $7a^3$, which is 700%.

Problem 2. What is the largest number of pieces a ring can be divided into using three straight lines?



Result. 9

Solution outline. We want all intersections of the lines to be distinct, lie in the ring and every line to be a secant of the inner circle.



Problem 3. A five digit number $a679b$ is divisible by 72. Find the value of $a \cdot b$.

Result. $3 \times 2 = 6$

Solution outline. Since $a679b$ is divisible by 8, it follows $b = 2$. Since it is divisible by 9, $9 \mid a + 6 + 7 + 9 + 2$ implying that $a = 3$.

Problem 4. A math teacher decided to organize two rounds of math competition. Each team consisted of five members. In the first round, the students divided themselves into teams on their own. In the second round the teacher divided them so that nobody was in the same team with anyone he or she has played with in the first round. Determine the minimum number of students in which such division is possible.

Result. 25

Solution outline. The number of students is divisible by 5 and is larger than 20 (Pigeonhole principle). It is easy to verify that 25 is sufficient.

Problem 5. The whole surface of a rectangular prism-shaped vanilla cake with edges of lengths 10, 10 and 5 is covered by a thin layer of chocolate. Let us cut the cake into cubes of volume 1. What percentage of cubes have no chocolate on them?

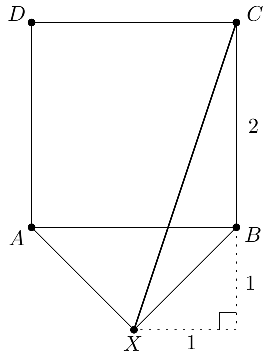
Result. $38.4\% = 48/125$

Solution outline. Consider what is left after eating all the pieces with chocolate layer. It will be a rectangular prism $8 \times 8 \times 3$ out of all pieces without chocolate. There must be then 192 pieces of cake without chocolate on it. From the total amount of pieces with chocolate $10 \cdot 10 \cdot 5 = 500$ we easily count the desired percentage: $100 \cdot (192/500)\%$.

Problem 6. We are given a square $ABCD$ with side length 2 and a point X in its plane (outside the square) so that $|AX| = |XB| = \sqrt{2}$. What is the length of the largest diagonal in the pentagon $AXBCD$?

Result. $\sqrt{10}$

Solution outline. We aim to find CX. Let E be the foot of perpendicular dropped from X onto BC. As $\triangle AXB$ is right and isosceles we have $EB = EX = 1$ and the rest follows from Pythagorean theorem.



Problem 7. Jacob had written $5 \cdot 414 = 1121$ on a blackboard. Now he is wondering how to increase or decrease each of the digits by 1 so that the result is correct. What will be the right-hand side after the change?

Result. 2012

Solution outline. It suffices to go through 16 ways to vary the left-hand side and see if the right-hand side can be changed accordingly. The process can be speeded up by various observations (for instance the result has to start with 2, hence the first number is 4 and the second one starts with 5).

Problem 8. The sum of integers x and y is at most 200 and their difference is smaller than 100. Find the maximum value of the expression $2 \cdot \min(x, y) + \max(x, y)$.

Result. 300

Solution outline. Without loss of generality let $x \geq y$. Then $2 \cdot \min(x, y) + \max(x, y) = 2y + x = y + (x + y) \leq y + 200 \leq \frac{x+y}{2} + 200 = 300$. This value is achieved for $x = y = 100$.

Problem 9. Let x, y, z be real numbers such that the arithmetic mean of x and $2y$ is equal to 7, and arithmetic mean of x and $2z$ is equal to 8. What is the arithmetic mean of x, y, and z?

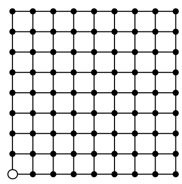
Result. 5

Solution outline. Adding the given equations we get

$$15 = \frac{x + 2y}{2} + \frac{x + 2z}{2} = x + y + z.$$

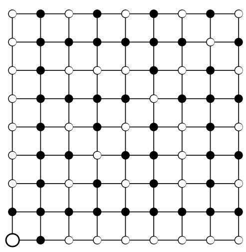
Divide by 3 to get the result.

Problem 10. 81 trees grow on a 8×8 square grid. The gardener cut out one of the corner trees and is now standing at that corner facing the rest of the grid. However, he does not see some of the trees because they are aligned with the other trees. (A tree T is aligned with another tree if there exists a tree on the line segment between the gardener and T.) How many trees does the gardener see?



Result. 45

Solution outline. Let us order the trees by their distance to the gardener and check each one for visibility. If a tree is not crossed out, we mark it as visible and cross out all the trees aligned with it.



Problem 11. How many ways are there to color the faces of a cube with two colors? Two colorings are considered identical if we can get one from the other by rotating the cube.

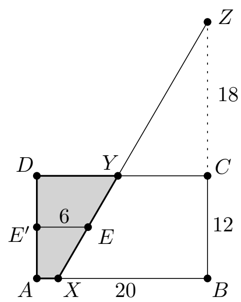
Result. $1+1+2+2+2+1+1=10$

Solution outline. We shall distinguish the cases according to the number of black faces involved in the coloring. For zero black faces we have one option as well as for one black face. For two black faces there are two ways—either the faces are neighbouring or they are not. For three black faces there are again two colorings (either these three faces share a common vertex or they do not). For four, five, and six the situation is analogous to the one with two, one, and zero black faces, respectively (we can place the white faces instead of the black ones). Altogether, we have $1+1+2+2+2+1+1=10$ ways to color the cube.

Problem 12. A rectangle $ABCD$ is given with $AB=20$ and $BC=12$. Let Z be the point on the ray BC such that $CZ=18$ and E the point inside $ABCD$ such that the distance from E to both AB and AD is 6. Let line EZ intersect AB and CD at points X and Y, respectively. Find the area of $AXYD$.

Result. 72

Solution outline. Focus on trapezoid $AXYD$ and observe that E is the midpoint of its side XY. Hence if we denote the foot of perpendicular dropped from E onto AD by E' then EE' is the midline of the trapezoid $AXYD$ and its area is thus $AD \cdot EE' = 72$.



Problem 13. For how many positive integers a ($1 \leq a \leq 2012$) is a^a a square of a positive integer?

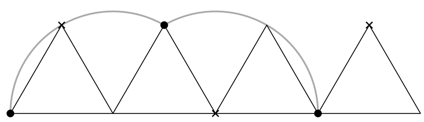
Result. 1028

Solution outline. If $a=2k$ is even then $a^a=(a^k)^2$ is always a square. If a is odd then a^a is a square if and only if a itself is a square. As $44^2 < 2012 < 45^2$ the answer is $\frac{1}{2}2012 + \frac{1}{2}44 = 1028$.

Problem 14. An equilateral triangle with side length 1 is lying on the floor, with one altitude perpendicular to the floor. We color one of its vertices red and we "roll" the triangle on the floor (in the plane of the triangle) through one full rotation. What is the length of the red vertex's trajectory?

Result. $\frac{2}{3} \cdot 2\pi = \frac{4}{3}\pi$

Solution outline. Observe that the red vertex traces circular arcs. Twice it traces $\frac{1}{3}$ of a circumference of a circle with radius 1 and once it remains fixed. Hence the total length of its trajectory equals $(\frac{1}{3}+\frac{1}{3}) \cdot 2\pi \cdot 1 = \frac{4}{3}\pi$.



Problem 15. What is the smallest positive integer consisting only of the digits 0 and 1 that is divisible by 225?

Result. 11111111100

Solution outline. The number has to be divisible by 25 so it ends with two zeroes. Moreover, it has to be divisible by 9 ($225 = 9 \cdot 25$), so the number of 1's is a multiple of nine. The smallest positive integer with these properties is 11111111100.

Problem 16. Bill is old enough to vote but not old enough to use the senior discount. (His age is between 18 and 70.) It is known that x years ago, the square of his age was the same as his current age increased by x. Moreover, Bill's age is a square of an integer. Find x.

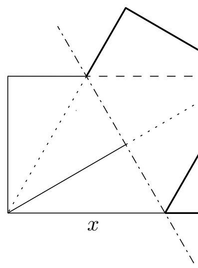
Result. 28

Solution outline. Bill's age a is the arithmetic mean of the positive integer $a - x$ and its square $a + x$. Therefore all possible values of a are $\frac{1}{2}(6 + 36) = 21$, $\frac{1}{2}(7 + 49) = 28$, $\frac{1}{2}(8 + 64) = 36$, $\frac{1}{2}(9 + 81) = 45$, $\frac{1}{2}(10 + 100) = 55$, and $\frac{1}{2}(11 + 121) = 66$. The only square among them is 36, hence $x = 36 - 8 = 8^2 - 36 = 28$.

Problem 17. Let us fold the bottom left corner of a rectangular paper to its top right corner. The resulting figure consists of three triangles created by the edges of the paper and the fold. For what ratio of side lengths of the paper is the ratio of the areas of the triangles $1 : 2 : 1$?

Result. $\sqrt{3} : 3 = 1 : \sqrt{3}$ or in reverse order.

Solution outline. For the two corners to coincide we have to fold about the line passing through the center of the rectangle perpendicular to one of its diagonals.



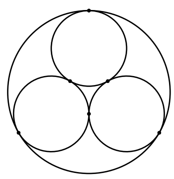
Focus on the resulting pentagon consisting of two congruent right triangles and an isosceles triangle in the middle. The diagonal splits the isosceles triangle into two congruent right triangles. Using the ratio of the areas we learn that in fact all four right triangles forming the pentagon are congruent. Hence the angle by their common vertex is $\frac{1}{3} \cdot 90° = 30°$. The ratio of the side lengths of the rectangle is readily obtained invoking the symmetry about the fold line.

Problem 18. How many three digit numbers are divisible by 6 if each digit is larger than 4?

Result. 16

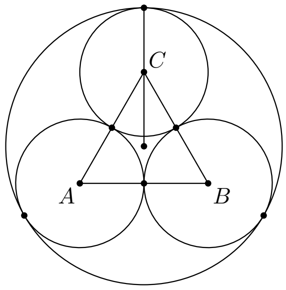
Solution outline. The number has to be divisible by 2, thus it must end with 8 or 6. Also, it has to be divisible by 3, so its digit sum has to be divisible by 3, which implies that after dividing by 3 the sum of the first two digits gives remainder 1 (if the last digit is 8) or 0 (if the last digit is 6). The remainder 1 can be achieved as a sum of numbers with remainders $2 + 2$, $3 + 1$ (the order of the digits matters, so we have $4 + 4 = 8$ combinations). Similarly the remainder 0 can be achieved only as $0 + 0$, $2 + 1$ (again $4 + 4 = 8$ combinations).

Problem 19. Three circles with radius 1 are given, such that each two are externally tangent. We put all the circles into a greater circle ω. Each small circle is internally tangent to the greater circle. Find the radius of the circle ω.



Result. $1 + \frac{2}{\sqrt{3}} = 1 + \frac{2\sqrt{3}}{3}$

Solution outline. Denote by A, B, C the centers of the smaller circles and by S the center of ω. Observe that A, B, C are vertices of an equilateral triangle with side length 2. Its altitudes are of length $\sqrt{2^2 - 1} = \sqrt{3}$, and they intersect at S. However, in equilateral triangle the altitudes are at the same time the medians. Hence $AS = \frac{2}{3} \cdot \sqrt{3}$. The radius of ω is then the sum of AS and the radius of the smaller circle.



Problem 20. Let n be a positive integer. If n^2 has 7 in the tens place, which digits can be in the units place?

Result. 6

Solution outline. Let $n = 10x + y$ for an integer $x \geq 0$ and a digit y. Then $n^2 = 100x^2 + 20xy + y^2$. The digit in the tens place is odd if and only if the digit in the tens place of the number y^2 is odd. Therefore y^2 is 16 or 36. The last digit in both cases is 6.

Problem 21. If we write the numbers $1, 2, \ldots, n$ in some order, we will get an n-chain. For example, one possible 11-chain is

$$3764581121910.$$

What is the smallest n with $n > 1$ such that there exists an n-chain that is a palindrome? (A number is a palindrome if it may be read the same way in either direction.)

Result. 19

Solution outline. The following is a palindromic 19-chain:

$$9|18|7|16|5|14|3|12|1|10|11|2|13|4|15|6|17|8|19.$$

We will show that 19 is the smallest possible value of n. First note that only one digit can appear in a palindromic chain odd number of times (namely the middle one). Clearly, for $n \leq 9$ this condition cannot be satisfied. Similarly, for $10 \leq n \leq 18$ both digits 0 and 9 appear exactly once, thus such an n-chain cannot be a palindrome.

Problem 22. Find all triples (x, y, z) of positive real numbers for which $(x+y)(x+y+z) = 120$, $(y+z)(x+y+z) = 96$, $(z+x)(x+y+z) = 72$.

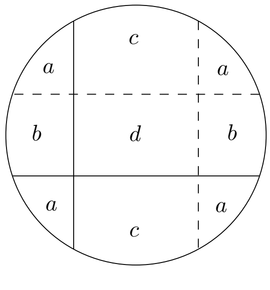
Result. $(4, 6, 2)$

Solution outline. Adding the equations yields $(x+y+z)(z+y+x+z+x+y) = 288$. Since we are looking for x, y, z positive, after dividing by 2 and taking the square root, we have $x+y+z = 12$. By substituting back to the original equations we get $(12-z) = 10$, $(12-x) = 8$, $(12-y) = 6$, from which we easily obtain the solution $(x, y, z) = (4, 6, 2)$.

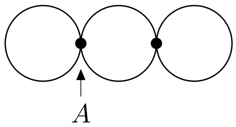
Problem 23. A circle is given with a radius 1 and two perpendicular chords inside it, dividing the circle into 4 parts. We color the part with the greatest and the part with the smallest area in black and leave the rest white. We know that the area of the white parts is the same as the area of the black parts. What is the maximum possible distance of the longer chord to the centre?

Result. 0

Solution outline. Draw the chords symmetrical to the two we already have about the center of the circle and observe that if the two areas are to be the same then one of the chords has to be a diameter of the circle.



Problem 24. An officer's route consists of three circles (shown in the picture). He must start at A and travel the entire route, without visiting any portion twice except perhaps the intersection points, and return to A. If his direction matters, how many such routes exist?



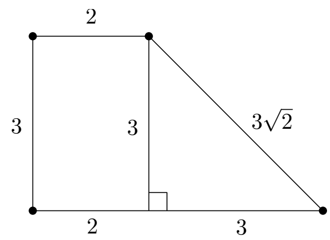
Result. 16

Solution outline. The officer has to walk around the first building either in the very beginning or in the very end of his route (in any of the two possible directions) which gives us 4 options. For any of the remaining two buildings he may independently decide if he walks around them clockwise or counterclockwise. These decisions determine the route uniquely, hence the answer is $4 \cdot 2 \cdot 2 = 16$.

Problem 25. A trapezoid has bases of lengths 5 and 2 and legs of lengths $3\sqrt{2}$ and 3. What is its area?

Result. $\frac{21}{2}$

Solution outline. A line parallel to the shorter leg passing through another vertex of the trapezoid splits it into a parallelogram and a triangle with side lengths 3, $5 - 2 = 3$, and $3\sqrt{2}$, which is therefore isosceles and right. The area is now simply $\frac{1}{2} \cdot 3 \cdot 3 + 3 \cdot 2 = \frac{21}{2}$.



Problem 26. There are 42 people in a row. They want to order themselves according to their height, so that the tallest one will stand in front. In one step, two people who are next to each other can switch position. At most, how many steps are necessary for them to order as they want?

Result. $\frac{n(n-1)}{2} = 21 \cdot 41 = 861$

Solution outline. Let us assign to each ordering a value H that denotes the number of (not necessary neighbouring) pairs such that the taller person stands behind the shorter one. Every step decreases H by at most one and if $H > 0$ then there exists a step that decreases it. In the beginning, H is at most $\frac{1}{2} \cdot 41 \cdot 42$ (if every pair is switched, i.e. if the people are in the opposite order). Thus, the "worst" ordering requires 861 steps.

Problem 27. Parsley lives in Vegetable State where one can pay only by coins with values 7 and 11. If Parsley had an unlimited supply of both kinds of coins what would be the highest integer price he could not pay with them?

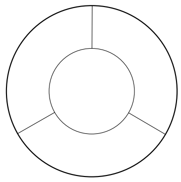
Result. 59

Solution outline. If he can pay some price, he can also pay this price plus a multiple of 7. Thus, it is enough to observe the multiples of 11 and their remainders modulo 7. $0 \bmod 7 = 0$, $11 \bmod 7 = 4$, $22 \bmod 7 = 8$, $33 \bmod 7 = 5$, $44 \bmod 7 = 2$, $55 \bmod 7 = 6$, $66 \bmod 7 = 3$. From 66 on, he can pay all the amounts, as the remainders repeat themselves. Thus, the last price he cannot pay is $66 - 7$. (Should he be able to pay this amount, it would hold $66 - 7 = 11k + 7m$. The left-hand side modulo 7 is 3, while the right-hand side is not 3 as the first multiple of 11 ($11k$) with remainder 3 is 66 as shown above.)

Problem 28. Let us divide a circle with radius 1 into 4 parts. What is the smallest possible perimeter of the part with the greatest area? If there is more than one part with the greatest area, we take into account the one with the smallest perimeter.

Result. π

Solution outline. A shape with a given area has the smallest perimeter if it is a circle. So all that needs to be done is to cut out the circle with area consisting of one fourth of the big circle and dividing the rest into three parts with equal area.



Problem 29. Find the sum of all real numbers a for which the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have at least one common real root.

Result. -2

Solution outline. Let x be a common solution of both equations. By subtracting the equations we get $(a-1)(x-1) = 0$. So either $x = 1$ or $a = 1$. If $x = 1$, $a = -2$ (substitute back). In the latter case the equations do not have any real roots.

Problem 30. How many 8-digit numbers are there such that after crossing out its first digit (from the left), one gets a number 35 times smaller than the original one?

Result. 0

Solution outline. Let m be a number that we get after crossing out the first digit and c be the crossed out digit. The original number is then $c \cdot 10^7 + m$. It should hold that $m = \frac{1}{35}(c \cdot 10^7 + m)$, which can be simplified to $17m = c2^6 5^7$. However, the left-hand side is nonzero and divisible by 17, while the right-hand side is not for $c \leq 9$.

Problem 31. Consider a right triangle with sides of integer length. One of the sides has length 2012. What is the maximum area it can have?

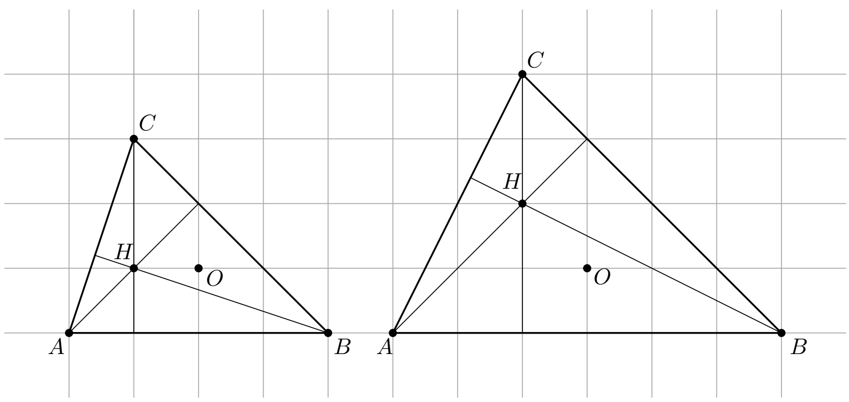
Result. $1006 \cdot (1006^2 - 1) = 1018107210$

Solution outline. In order to maximize the area, the side of length 2012 is surely the shorter leg. From the Pythagorean theorem we get $b^2 + 2012^2 = c^2$. We want to maximize $2012 \cdot b/2$. That is the same as to minimize the difference between the lengths of the hypotenuse and the longer leg. If $c = b + 1$, the parity does not match. If $c = b + 2$, we get a simple equation for b.

Problem 32. Let ABC be a triangle with circumcentre O and orthocentre H in the Cartesian plane. No two of these five points coincide and all of them have integer coordinates. What is the second smallest possible radius of a circle circumscribed around triangle ABC?

Result. $\sqrt{10}$

Solution outline. Consider the possible radii of the excircle. For 1, $\sqrt{2}$, 2 and $2\sqrt{2}$ we see that the orthocentre coincides with one of the vertices. For $\sqrt{5}$ and $\sqrt{10}$ we can construct the desired triangles.



Problem 33. Find the largest natural number n such that the number $7^{2048} - 1$ is divisible by 2^n.

Result. 14

Solution outline. Use the formula $a^2 - b^2 = (a-b)(a+b)$ several times to get $7^{2048} - 1 = (7-1)(7+1)(7^2+1)(7^4+1)\dots(7^{1024}+1)$. But for the $(7+1)$, all the other factors in the product are divisible by 2 exactly once, since for every positive integer k, $7^{2k} + 1$ gives a remainder 2 when divided by 4.

Problem 34. If we calculate the product of the digits of some number, the product of the digits of the product, and so on, we arrive after a finite number of steps at a one digit number. The number of required steps is called the *persistence* of the number. For example, the number 723 has persistence 2 because $7 \cdot 2 \cdot 3 = 42$ then $4 \cdot 2 = 8$. Find the greatest even number with mutually distinct digits that has persistence 3.

Result. 98764312

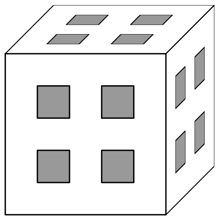
Solution outline. Let us observe the highest number that satisfies the condition of different nonzero digits which is 987654321, but it has persistence only 2 because there is a five and an even number, so that in the second step we obtain 0. To obtain the result, it suffices to get rid of 5, and switch the last two digits to satisfy the evenness.

Problem 35. If we extend the sides AD and BC of a convex quadrilateral, they intersect at a point E. Let us denote H and G the midpoints of BD and AC, respectively. Find the ratio between the area of a triangle EGH and the area of a quadrilateral $ABCD$. (We let you know that this ratio is the same for all convex quadrilaterals with non-parallel sides.)

Result. $1:4$

Solution outline. Let us consider a quadrilateral in which points C and D merge into one point. Then GH is a midline in ABC and the ratio is obviously $\frac{1}{4}$.

Problem 36. Cube termites bore four straight square tunnels of sidelength 1 in each direction inside a cube (as you can see on the picture) and now they have left the cube. How many cm^2 of paint do we need to cover the surface of what is left of the cube if the original cube had side length 5 cm?



Result. 270

Solution outline. Let us imagine that we are looking at the cube from above and count the area of the surfaces oriented upwards. In the depth of 0 cm (the side of the cube) there are 21 cm^2, in the depth of both 2 and 4 cm there are 12 cm^2. The situation is the same from all directions.

Problem 37. We are given a circle with radius 1. We are standing on the leftmost point of the circle. It is possible to move only up and right. What is the length of the longest trajectory that we can travel inside the circle?

Result. $1+\sqrt{2}$

Solution outline. For each trajectory there exist a trajectory of the same length that first moves only right and then only upwards, so it suffices to consider such trajectories only. The longest trajectory must clearly pass through the centre. Let us denote a the length of the trajectory travelled to the right from the centre of the circle and b the length of the trajectory that we will travel from the centre upwards. As the longest route will end at the circumference of the circle, it holds that $a^2+b^2=1$. We want to maximize $a+b$, which is equivalent to maximizing $(a+b)^2 = a^2+b^2+2ab = 1+2ab$. From the trivial inequality $0 \le (a-b)^2 = a^2-2ab+b^2$ we have $2ab \le a^2+b^2 = 1$. The equality occurs when we have equality for $0 \le (a-b)^2$, so $a = b = \sqrt{1/2} = \frac{\sqrt{2}}{2}$. Adding the distance to the centre of the circle, we get the result $1+\sqrt{2}$.

Problem 38. What is the greatest divisor of $15! = 1 \cdot 2 \cdot \ldots \cdot 15$ that gives a remainder of 5 when divided by 6?

Result. $5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 13 = 175175$

Solution outline. The remainder of the product is the same as the product of the remainders of the factors, for example $7! \bmod 6 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 0 \cdot 1 = 0$. In order to make the final remainder equal to 5, the number can be divisible by neither 2 nor 3, so we have to exclude all the two's and three's from 15!. After that we are left with the number $5^3 \cdot 7^2 \cdot 11 \cdot 13$, which gives the remainder 1. Each factor gives a remainder either 1 or 5, so we just have to exclude one number with remainder 5. The smallest such number is 5.

Problem 39. A cube and 27 of its points are given: the vertices, midpoints of the edges, centers of the faces, and the center of the cube. How many lines are passing through exactly three of the given points?

Result. 49

Solution outline. We divide the lines into three groups: those that pass through the center of the cube ($9+8/2=13$), those that pass through the center of the side but not the center of the cube (4 for each side), and those lines on which the edges lie (12). We conclude that there are $13+24+12$ lines altogether.

Problem 40. n participants took part in a competition that lasted for k days. Each day each participant received an integer amount of points between 1 and n, inclusive. No two participants had the same score in a given day. At the end of the competition (the k-th evening) each participant had a score of 26 points (when all the points for the whole competition were added). Find the sum of all n for which this is possible (regardless of k).

Result. $1+3+12+25=41$

Solution outline. It is clear that $n<26$. Altogether, there were $\frac{n(n+1)}{2}$ points allocated each day, that gives a total sum of $k\cdot\frac{n(n+1)}{2}$ points. However, that is also equal to $26n$. Comparing these two expressions, we get $k(n+1)=52=2\cdot2\cdot13$. Trying out several possibilities, we find out that for $k\in\{2,4,13,26\}$ we can find a way in which the contestants could receive points during the competition and for $k\in\{1,52\}$ we cannot find such a way. The sum of all n is then $(26-1)+(13-1)+(4-1)+(2-1)$.

Problem 41. We want to cut a cylindrical cake with 5 straight cuts. What is the maximum number of resulting pieces? For example, by 3 cuts we can divide the cake into 8 pieces.

Result. 26

Solution outline. For the sake of simplicity, consider an infinite 3-dimensional space that we cut by planes. When the space is cut in some way, the next plane adds as many new parts as it had cut through. Let us take the lines that are intersections of the older planes with the new plane. The number of the new pieces is the same as the number of the parts that the lines are dividing the new plane into. Thus, $v_n=v_{n-1}+p_{n-1}$, where v_n is the result for n intersections and p_n is the number of new parts, i.e. the number of parts into which we can divide a plane by lines. By drawing on paper we can find the maximum values for p_1, p_2, p_3 and p_4, which are 2, 4, 7 a 11. For $v_1=2$ we get the result $2+2+4+7+11=26$.

For curious souls, this is a generalization: $\binom{n}{3}+\binom{n}{2}+\binom{n}{1}+\binom{n}{0}=1/6(n^3+5n+6)$.

Problem 42. An eight-branched star (Stella octangula) is a solid, which results from sticking eight regular tetrahedrons to the faces of an octahedron. All the edges of each tetrahedron and the octahedron have length 1. What is the volume of the eight-branched star?

Result. $\sqrt{2}$

Solution outline. We have two types of solids: A regular tetrahedron with side length 1 and a tetrahedral pyramid with all sides of length 1. The base area of the tetrahedral pyramid is 1 and the height can be obtained from Pythagorean theorem as $\sqrt{1^2-\left(\frac{\sqrt{2}}{2}\right)^2}=\frac{1}{\sqrt{2}}$, so its volume is $\frac{1}{3\cdot\sqrt{2}}$. Similarly, the base area of the tetrahedron (equilateral triangle) is $\frac{\sqrt{3}}{4}$ and the height $\sqrt{1^2-\left(\frac{2\sqrt{3}}{6}\right)^2}=\frac{\sqrt{2}}{\sqrt{3}}$, so its volume is $\frac{\sqrt{2}}{3\cdot4}$. When we sum it up, we get $2\cdot\frac{\sqrt{2}}{6}+8\cdot\frac{\sqrt{2}}{12}=\sqrt{2}$.

Other solution: Watch what happens if you inscribe the star into a cube.

Problem 43. A hitchhiker is walking along the road. The probability that a car picks him up in the next 20 minutes is $\frac{609}{625}$. What is the probability of a car picking him up in the next five minutes, if the probability that he gets picked up by a car is the same in each moment?

Result. $3/5$

Solution outline. The probability that the hitchhiker is not picked during the first 20 minutes is $1-\frac{609}{625}=\frac{16}{625}$. If the probability of him not being picked during 5 minutes is p, then for the 20 minutes it is p^4. Hence $p=2/5$ and the probability of catching a car is $1-2/5$.

Problem 44. A vandal and a moderator are editing a Wikipedia article. At the beginning, the article was without a mistake and each day the vandal adds one mistake. At the end of each day the moderator has $\frac{2}{3}$ chance of having found each single mistake that is in the article. What is the probability that after three days the article will be without a mistake?

Result. $2/3\cdot8/9\cdot26/27=\frac{2^5\cdot13}{3^6}$

Solution outline. For each mistake we find the probability that it won't stay till the end. The probability that the mistake will stay k days is $(1/3)^k$, and the probability that some mistake will not stay k days is $1-(1/3)^k$. The events of not spotting a mistake are independent, so we can multiply: $(1-1/3)(1-1/9)(1-1/27)$.

Problem 45. We have a large enough heap of red, blue, and yellow cards. We can receive the following number of points:

- for each red card one point,
- for each blue card twice the number of red cards as points,
- for each yellow card three times the number of blue cards as points.

What is the maximum number of points we can receive when we have 15 cards?

Result. 168

Solution outline. Let R denote the number of red cards, B the number of blue cards and Y the number of yellow cards. Each red card adds $1 + 2B$ points to the overall score (1 for itself and 2 for each blue card) and each yellow card adds $3BY$ points. Therefore $R + 2RB + 3BY$ is the overall score. For $B = 0$, the maximum score is 15. For $B = 1$, the score is always $R + 2R + 3Y = 3(R + Y) = 42$. For $B > 1$, it is worth to swap all red cards for yellow cards. Therefore we choose $R = 0$ for $B > 1$ and get the overall score $3BY$ in this case. Respecting the condition $B + Y = 15$, we get the maximum at $B = 7$ and $Y = 8$, which yields the score 168.

Problem 46. Matthew has one 20-sided die and his friend CD has three 6-sided dice. What is the chance that after rolling all the dice, the value on Matthew's die will be greater than the sum of the values on CD's dice?

Result. $\frac{19}{40}$

Solution outline. First observe that both rolls have a symmetric distribution, i.e., the probability that we get x on a 20-sided die is the same as the probability that we get $21 - x$ (from 1 to 20). Similarly, the probability that we get y on three 6-sided dice is the same as the probability that we get $21 - y$ (from 3 to 18). By a similar line of thought we may conclude that the probability of Matthew winning is the same as the probability of CD winning. Thus, the answer is $\frac{1-p}{2}$, where p is the probability of a draw. And that is $1/20$, because whatever CD rolls, Matthew always has a $1/20$ chance that he rolls the same.

Problem 47. Let us have a 10 by 10 board. The rows and columns are numbered from left to right and top to bottom, respectively, by integers from 1 to 10. In each cell we write the product of the row number and column number. A traveler stands on the top left cell and wants to arrive to the bottom right cell. However, she can only travel right and down (not diagonally). A *traveler's number* is the product of the numbers on the cells which she had stepped onto (including the first and the last one). What is the greatest common divisor of all possible traveler's numbers?

Result. $10 \times 10! \cdot 10! = 2^{17} \cdot 3^8 \cdot 5^5 \cdot 7^2$

Solution outline. During each route, the traveler gathers each of the numbers 2, 3, ..., 9 at least twice (one time for the column and one time for the row). Furthermore, for each i except for 1 and 10, there exists a route that crosses the i-th row exactly once and i-th column exactly once. Thus, the $2, 3, \dots, 9$ will occur in the result exactly once. As for the 10, each traveler's number contains at least three 10's. (The traveler must step on the bottom right corner and on the previous where there is 10, too.) It is easy to verify that there is a route that does not contain more than three 10's.

Problem 48. We have a triangle with altitudes of sizes 3, 4, and 6. What is its perimeter?

Result. $\frac{72}{\sqrt{15}} = \frac{24\sqrt{15}}{5}$

Solution outline. From the fact that the area of the triangle is half of the product of its height and length of its side we get that the triangle will be similar to the triangle with sides 2, 3, 4. We can express one of the altitudes using its sides. For example, we can use the Heron's formula for sides of length $2a$, $3a$ a $4a$ and then compare this with the area $\frac{2a \cdot 6}{2}$. Thus, the $A = \frac{a^2}{4}\sqrt{135} = 6a$, from which it follows that $a = \frac{24}{\sqrt{135}}$ and perimeter is then $9a$.

Problem 49. Find the greatest integer $n \leq 4\,000\,000$ for which $\sqrt{n + \sqrt{n + \sqrt{n + \dots}}}$ is rational.

Result. $1999 \cdot 2000 = 3998000$

Solution outline. Denote $s = \sqrt{n + \sqrt{n + \sqrt{n + \dots}}}$, then $s = \sqrt{n + s}$. By solving the quadratic equation for s we get $s = \frac{1 \pm \sqrt{1+4n}}{2}$, and because s is rational, also $1 + 4n = a^2$ for some a rational. After simple manipulation we have $n = \frac{a^2-1}{4}$. In order for n to be an integer, a has to be an integer as well: If it were $a = \frac{p}{q}$ with p, q coprime, then $n = \frac{p^2-q^2}{4q^2}$ and q would divide p, which is a contradiction. Since $n = \frac{a-1}{2} \cdot \frac{a+1}{2}$, a must be odd. Moreover, from $n \leq 4\,000\,000$ we have $\frac{a+1}{2} = 2000$. Thus, $n = 1999 \cdot 2000$ is the solution.

Problem 50. A 3×3 Maxi-square is a square divided into nine square tiles. Each tile is divided into four little squares in which the numbers 1, 2, 3, and 4 are written (each of them exactly once). Two tiles can touch only if their adjacent numbers match (as in dominoes). How many Maxi-squares exist?

4	3	3	4	4	2
1	2	2	1	1	3

• • •

Result. $24 \cdot 7 = 168$

Solution outline. First we choose the tile in the middle to be (1, 2 and 3, 4). Denote by A, B the numbers written in the little square at the coordinates $(4, 1)$ and $(4, 6)$, respectively ($(1, 1)$ being the upper left corner square). Observe that $A \in \{3, 4\}$ and $B \in \{1, 2\}$. If the couple (A, B) equals one of $(3, 2)$, $(4, 1)$, $(3, 1)$, there is only one way to fill the rest of the maxi-square. In the case $(A, B) = (4, 2)$, there are four ways. As there are $4! = 24$ possible middle tiles, the result is $24 \cdot 7$.

Problem 51. Andrew calls his favourite number *balloon.* It holds for balloon that

- the sum of its digits is twice the number of its digits
- it does not have more than 12 digits
- its digits are in turn even and odd (it doesn't have to start with an even digit)
- the number greater by one is divisible by 210.

Find the value of Andrew's balloon.

Result. 1010309

Solution outline. Let us focus on the condition that the number increased by 1 is divisible by $210 = 2 \cdot 3 \cdot 5 \cdot 7$. From this it immediately follows that the last digit has to be 9. Because of the divisibility by three, the digit sum of the balloon is of the form $3k + 2$ and also it should be twice the number of digits from the statement. So the digit sum is even, of the form $3k + 2$ and greater than 9, thus it can only be 14 or 20, to which correspond numbers with 7 and 10 digits, respectively. Since odd and even digits alternate, the smallest possible solutions relaxing the condition on divisibility by 7 are 1010109 and 2101010109 with digit sums 12 and 15. The digit sum in the second case is odd, so there is no way of achieving the digit sum of 20. In the first case we have to increase the digit sum by 2, which means that we have to increase one digit by 2. We can easily check that the only solution (taking divisibility by 7 back into consideration) is the number 1010309.

Problem 52. Find all four-digit positive integers n such that the last four digits of the number n^2 is the number n.

Result. 9376

Solution outline. We seek n such that $n^2 - n = n(n-1) = 10000x$ for some integer x. As 5^4 divides the right-hand side, it must divide the left-hand side; thus $5^4 = 625$ divides either n or $n-1$, since they are coprime. Similarly $2^4 = 16$ divides either n or $n - 1$. The former divisibility implies that n is of the form $625k + b$, where $k \in \{1, 2, \dots, 15\}$ and $b \in \{0, 1\}$. The remainder of 625 when divided by 16 is 1, therefore the remainder of n when divided by 16 is $k + b$. Hence $k + b \in \{0, 1, 16\}$ from the latter divisibility. It is then readily seen that n has 4 digits if and only if $k = 15$, $b = 1$, $n = 9376$; we can easily verify that this n satisfies all the conditions.

Problem 53. Find the sum of all five-digit palindromes.

Result. $900 \cdot 55000 = 49500000$.

Solution outline. We establish a one-to-one correspondence between 5-digit palindromes and 3-digit numbers—the number abc corresponds to the palindrome $abcba$ for a non-zero digit a and arbitrary digits b and c. The total sum of the palindromes can be then computed digit-wise; each digit a adds $a \cdot 10001$, each digit b adds $b \cdot 1010$ and each c adds $c \cdot 100$ to the total sum. Each possible value of a appears in the total sum 100 times and each possible value of b or c appears 90 times. The total sum is thus $(1+2+\cdots+9)(100)(10001)+(0+1+\cdots+9)(90)(1110) = 45004500+4495500 = 49500000$.

Problem 54. How many ordered quadruples of odd positive integers (a, b, c, d) satisfy $a + b + c + d = 98$?

Result. $\binom{50}{3} = 19600$

Solution outline. First we rewrite the problem to get rid of the condition of a, b, c, d being odd. Let $a = 2A + 1$, and likewise for b, c, d. Then there is a one-to-one correspondence between quadruples (a, b, c, d) of odd positive integers satisfying $a+b+c+d = 98$ and quadruples (A, B, C, D) of nonnegative integers satisfying $A+B+C+D = \frac{1}{2}(98-4) = 47$. The solution is now a standard exercise on combinations with repetitions allowed; the answer is $\binom{50}{3}$ (each quadruple (A, B, C, D) corresponds to a sequence of 47 "balls" and 3 "separators").

Problem 55. Find the only eleven-digit number such that

- it starts with a one
- when it it is written twice in a row, it is a perfect square.

Result. $(10^{11} + 1) \cdot 16/121 = 16 \cdot 826446281 = 13223140496$

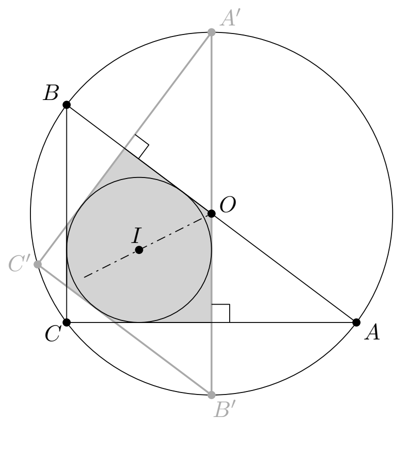
Solution outline. Let n be the number satisfying the conditions above. The number N created by subsequently writing n two times is equal to $n(10^{11} + 1) = n \cdot 11^2 \cdot D$ for a suitable integer D. Therefore, if we let $n = t^2 \cdot D$, we obtain $N = (11 \cdot t \cdot D)^2$, which is a perfect square. It remains to choose t in such a way that n has eleven digits and its decimal representation starts with 1. The only possible choice is $t = 4$.

Problem 56. Two different triangles with side lengths 18, 24, and 30 are given such that their incircles coincide and their circumcircles coincide. What is the area of the polygon that the triangles have in common?

Result. 132

Solution outline. Let ABC be one of the triangles ($AC = 24$, $BC = 18$). Let I be the incenter of ABC and O its circumcenter. Since $(18, 24, 30) = 6 \cdot (3, 4, 5)$, the angle ACB is right and the radii r and R of the inscribed and the circumscribed circle of ABC, respectively, are found as $r = \frac{1}{2}(18 + 24 - 30) = 6$ and $R = \frac{1}{2}30 = 15$. There is only one diameter $A'B'$ of the circle (ABC) different from AB that is tangent to the incircle of $\triangle ABC$; it is the one symmetric to AB with respect to the line OI. Note that $A'B'$ lies on the perpendicular bisector of AC. Indeed, this bisector passes through O, and since its distance from BC is 12, its distance from I is 6. The other triangle $A'B'C'$ is then symmetrical to ABC w.r.t. OI and $A'B' \perp AC$ and $A'C' \perp AB$. Finally, the intersection of our two triangles can be computed by taking the area of ABC and subtracting the areas of three small triangles similar to ABC; their side lengths are 9, 12, 15 (the triangle at A), 6, 8, 10 (at B), and 3, 4, 5 (at C), respectively. Thus the area of the intersection is

$$\frac{1}{2}(18 \cdot 24 - 9 \cdot 12 - 6 \cdot 8 - 3 \cdot 4) = 132.$$



Problem 57. In how many ways can we color the cells of a 5×5 grid with black and white so that in each column and row there will be exactly two black squares?

Result. 2040

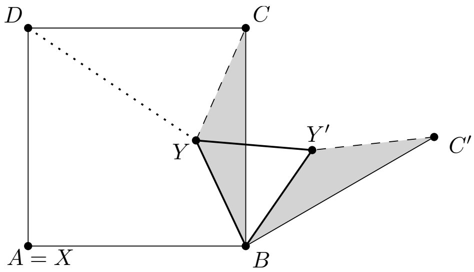
Solution outline. Let us paint the ten cells one after another. Denote the cells $A1, \ldots, E5$ (columns by letters). Without loss of generality, the cells $A1$, $B1$ are painted (from symmetry, it is sufficient to multiply the answer by $\binom{5}{2} = 10$) and also the cell $B2$ (multiply by four). If the cell $A2$ is painted, it remains to determine the number of complying paintings of 3×3 grid, which is 6. If a different cell in the second row is painted, w.l.o.g. let it be $C2$ (multiply by 3), and in the third column $C3$ (again multiply by 3). If $A3$ is now painted, there is only one possibility how to finish ($D4$, $D5$, $E4$, $E5$ are painted). Otherwise there are four possibilities (without loss of generality in the third row $D3$ is painted and $D4$ in the fourth column). To sum up, the overall number of colorings is

$$10 \cdot 4 \cdot \big(6 + 3 \cdot 3 \cdot (1 + 2 \cdot 2)\big) = 40 \cdot 51 = 2040.$$

Problem 58. Two distinct points X and Y lie inside a square with side length 1. By *remoteness* of a vertex of the square we mean its distance to the closer of the points X and Y. What is the smallest possible sum of the remotenesses of the vertices of the square?

Result. $\frac{\sqrt{6}+\sqrt{2}}{2}$

Solution outline. Let V be the value we are minimizing. If $X = A$ and $Y = B$ then $V = 2$ and using triangle inequalities it is easy to show that if X is closer to exactly 0, 2 or 4 vertices than Y, then $V \geq 2$. Next, let us assume that the remoteness of A is measured to X and the remotenesses of B, C, D are measured to Y, i.e., let us minimize the value $V = AX + BY + CY + DY$. The minimum is attained when $X = A$ and Y is the Fermat point of the triangle BCD, that is if $\angle BYC = \angle CYD = \angle DYB = 120°$. (The required property of the Fermat point can be verified by rotating the triangle BYC around B by $60°$ into a triangle $BY'C'$ and comparing the length of the broken line $DYY'C$ to the length of the segment DC'). Finally, we compute DC' using Pythagorean theorem. It turns out to be less than two so it is the answer.



Problem 59. The parking lot consists of 2012 parking places regularly distributed in a row with numbers 1 to 2012. 2012 cars park there one after each other, in the following way:

- The first car chooses randomly one of the 2012 places.
- The following cars choose the place from which the distance to the closest car is the greatest (each of such places with the same probability).

Determine the probability that the last car parks in the parking place number 1.

Result. $1/2012 \cdot 1/1025 = 1/2062300$

Solution outline. The place No. 1 can be taken as the last one only if place No. 2 is taken as the first one (with the probability of $\frac{1}{2012}$) and place No. 2012 right after it. Then some places from 3 to 2011 will be filled until gaps of size 1 and 2 remain. In that moment the probability of place No. 1 to be taken as the last one is equal to the inverse of number of the places, since every place has equal probability of being taken as the last one. Note that by looking only at sizes of the gaps and placing cars into the widest ones, we always end up with the same number of gaps of sizes 1 and 2. Thus we can uniquely determine their count.

Let $f(n)$ be the number of places which remain free, if the cars park on $n+2$ places in a row with first and last places already taken. The first car will park in the middle, which splits the task into the cases $f(\lfloor\frac{n-1}{2}\rfloor)$ and $f(\lceil\frac{n-1}{2}\rceil)$. Thus we have a recurrence relation $f(n) = f(\lfloor\frac{n-1}{2}\rfloor) + f(\lceil\frac{n-1}{2}\rceil)$ with initial conditions $f(1) = 1$ and $f(2) = 2$. After a bit of work and computing values for small n we can derive closed form:

$$f(x) = \begin{cases} x - 2^{n-1} + 1, & 2^n \leq x \leq \frac{3}{2} \cdot 2^n - 2 \\ 2^n, & \frac{3}{2} \cdot 2^n - 1 \leq x \leq 2 \cdot 2^n - 1. \end{cases}$$

Thus $f(2009) = 1024$ and the overall probability is $\frac{1}{2012} \cdot \frac{1}{1024+1} = \frac{1}{2062300}$.

Problem 60. Find all real numbers x that satisfy

$$(x^2+3x+2)(x^2-2x-1)(x^2-7x+12)+24=0.$$

Result. $0, 2, 1\pm\sqrt{6}, 1\pm\sqrt{8}$

Solution outline. Since

$$\begin{aligned}(x^2+3x+2)(x^2-7x+12) &= (x+1)(x+2)(x-3)(x-4) = \\ &= (x+1)(x-3)(x+2)(x-4) = (x^2-2x-3)(x^2-2x-8),\end{aligned}$$

after substitution $x^2-2x=z$ we get an equation

$$0=(z-3)(z-8)(z-1)+24=z^3-12z^2+35z=z(z-5)(z-7).$$

For each $z\in\{0,5,7\}$ we can easily determine the corresponding x.