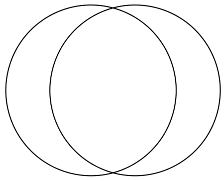
Problem 1J. Given that there is exactly one way to write 2013 as a sum of two primes find the product of these primes.

Result. 4022

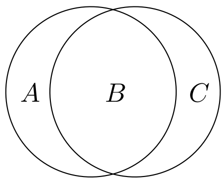
Solution outline. As the sum is odd, one of the primes must be even and therefore equal to 2. Thus $2013 = 2 + 2011$ is the only candidate which happens to work as 2011 is indeed a prime. The result then is $2 \cdot 2011 = 4022$.

Problem 2J. Two circles with radius 1 intersect each other. The area of their intersection equals the sum of the areas of the two outer parts. What is the area of the intersection?



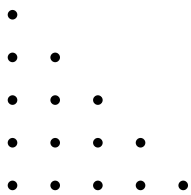
Result. $\frac{2}{3}\pi$

Solution outline. Denote the areas of the three parts as in the figure.



Combining the obvious $A = C$ (symmetry) with the given $B = A + C$ yields $B = 2A$. Thus the area of the intersection is equal to two thirds of the area of the left circle, that is $\frac{2}{3}\pi$.

Problem 3J. We have several pins five of which are yellow, four red, three green, two blue, and the remaining one is orange. In how many ways can we place them in the given triangular grid (see diagram) so that no pair of pins with the same color is in the same row or column? The pins of the same color are considered indistinguishable.



Result. 1

Solution outline. First we place the yellow pins and our only option is to place them on the hypotenuse of the triangle. By the same argument we now have an only option to place the four red pins (again on the hypotenuse of the triangle formed by empty spots). Continuing with this argument we see that there is a unique arrangement which satisfies the given conditions.

Problem 4J. Find the smallest positive integer whose product of digits equals 600.

Result. 3558

Solution outline. Since $600 = 2^3 \cdot 3 \cdot 5^2$, we may only use the digits 1, 2, 3, 4, 5, 6, 8. Using the digit 1 only increases the number, so we will do better without it. Also, we must use two fives as it is the only way to get 5^2. The product of the remaining digits then must be 24, thus one digit is not enough and from the pairs $3 \cdot 8$ and $4 \cdot 6$, the former contains the smallest digit and leads to the solution, which is 3558.

Problem 5J. Positive real numbers a and b satisfy

$$a + \frac{1}{b} = 7 \quad \text{and} \quad b + \frac{1}{a} = 5.$$

What is the value of $ab + \frac{1}{ab}$?

Result. 33

Solution outline. We multiply the two equations and obtain

$$\begin{aligned} ab + \frac{a}{a} + \frac{b}{b} + \frac{1}{ab} &= 35, \\ ab + \frac{1}{ab} &= 33. \end{aligned}$$

Problem 6J. Luke has a six-digit number which satisfies the following conditions:

1. The number reads the same from left to right as from right to left.
2. It is divisible by 9.
3. After crossing out its first and last digit the only prime factor of the resulting number is 11.

What is Luke's number?

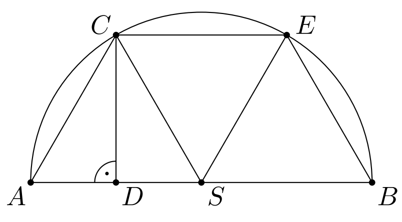
Result. 513315

Solution outline. We start by analyzing the last condition. The only four-digit power of 11 is 1331, therefore the sought number is of the form $\overline{a1331a}$ for some digit a. Since it is divisible by 9, so must be also its sum of digits $2a + 8$. Since $2a + 8$ is even and less than or equal to 26, the only choice is $2a + 8 = 18$, i.e. $a = 5$ and the answer is 513315.

Problem 7J. Point D lies on the diameter AB of the semicircle k. A perpendicular to AB through point D intersects k at point C. The lengths of the arcs AC and CB of k are in the ratio $1 : 2$. Find $AD : DB$.

Result. $1 : 3$

Solution outline. Since C trisects the arc AB, we can draw point $E \in k$ such that A, C, E, and B (in this order) are consecutive vertices of a regular hexagon.



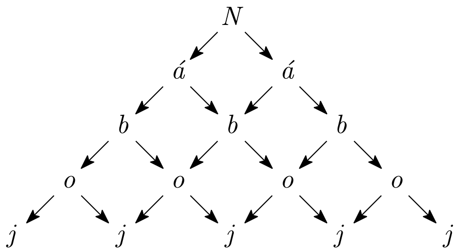
Now if S is the center of k, triangle ASC is equilateral with D the midpoint of SA (altitudes coincide with medians in equilateral triangles). Then we easily obtain $AD : DB = 1 : 3$.

Problem 8J. Two spoilt brothers Jim and Tim got a bag of chocolate chips and split them evenly. Both of them eat from two to three chocolate chips a day. Jim finished his chips after fourteen days, Tim after exactly three weeks. How many chocolate chips were there in the bag?

Result. 84

Solution outline. Note that Jim ate at most $3 \cdot 14 = 42$ chips while Tim ate at least $2 \cdot 21 = 42$. Since they both started with the same amount of chocolate chips, it must have been 42, and the answer is $42 + 42 = 84$.

Problem 9J. In how many ways can we read the word *Náboj* in the diagram?



Result. 16

Solution outline. From each of the letters *N*, *á*, *b*, *o* we can continue reading in exactly two ways. Therefore, we can read the word *Náboj* in $2^4 = 16$ ways.

Problem 10J. Inhabitants of an island are either liars or truth-tellers. Twelve of them sit around a table. They all claim to be truth-tellers and also claim that the person to their right is a liar. What is the maximum possible number of liars in such a group?

Result. 6

Solution outline. If two truth-tellers sat next to one another, the one on the left would not call his right neighbour a liar. For the same reason two liars cannot sit side by side. It remains to check that an arrangement in which truth-tellers and liars alternate satisfies the condition, making the number 6 our answer.

Problem 11J / 1S. Jane has 11 congruent square tiles six of which are red, three blue, and two green. In how many ways can she construct a 3×3 square from 9 of the 11 tiles such that the coloring of the square remains intact if we rotate the square by 90° clockwise around its center? Two tiles of the same color are considered indistinguishable.

Result. 0

Solution outline. For a coloring to be invariant under the 90° rotation, its four corner tiles must have the same color and the same holds for the remaining four non-central tiles. Thus we either need eight tiles of one color or two quadruplets of tiles with the same color. Since we have neither of that available, the answer is 0.

Problem 12J / 2S. Two fifths of males and three fifths of females living on a certain island are married. What percentage of the island's inhabitants is married?

Result. $48\% = \frac{12}{25}$

Solution outline. Denote by D the total number of married couples on the island. Then the total numbers of males and females are $\frac{5}{2}D$, $\frac{5}{3}D$, respectively. Thus, we have $\frac{5}{2}D + \frac{5}{3}D = \frac{25}{6}D$ inhabitants from which $2D$ are married. The fraction of married islanders is then

$$\frac{2D}{\frac{25}{6}D} = \frac{12}{25} = 48\%.$$

Problem 13J / 3S. What is the maximum side length of an equilateral triangle which can be cut from a rectangular-shaped paper with dimensions $21 \times 29.7\,\mathrm{cm}$?

Result. $14\sqrt{3} = \frac{42}{\sqrt{3}}\,\mathrm{cm}$

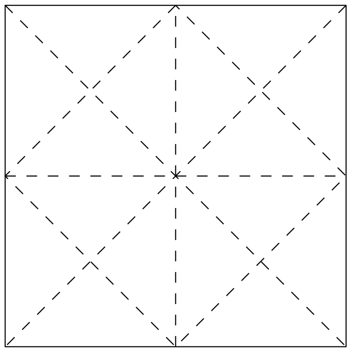
Solution outline. Any equilateral triangle lying between a pair of parallel lines can be suitably enlarged so that two of its vertices lie on opposite borderlines. From all such equilateral triangles the longest side length is obtained from one with the whole edge on one of the borderlines.

Then if the width of the strip is 21 cm, the largest fitting equilateral triangle has altitude 21 cm. The side-length is then $\frac{21\,\mathrm{cm}}{\sin 60^\circ} = 14\sqrt{3}\,\mathrm{cm}$. Since $14\sqrt{3} < 14 \cdot 2 < 29.7$, this triangle fits inside the rectangle.

Problem 14J / 4S. Kate took a square piece of paper and repeatedly folded it in half until it was done four times. After every fold, she was left with an isosceles right triangle. How many squares can one see after unfolding the paper?

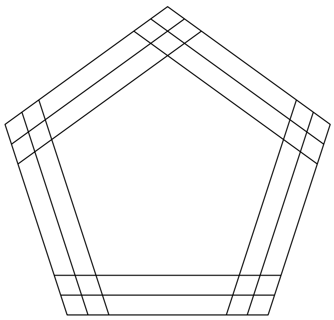
Result. 10

Solution outline. The fold-lines are captured in the following figure.



Altogether, we see 10 squares: the whole paper, the square connecting the midpoints of the edges and in each of these two, we see four smaller squares.

Problem 15J / 5S. How many pentagons are there in the figure?



Result. $3^5 = 243$

Solution outline. Observe that each pentagon must have the center of the figure in its interior. We have three choices for each of the five sides (outer, middle or inner line), thus there are $3^5 = 243$ pentagons.

Problem 16J / 6S. Joseph wanted to add two positive integers but accidentally wrote a zero digit to the end of one of them. The result he obtained was 3858 instead of the correct 2013. Find the value of the larger summand.

Result. 1808

Solution outline. Denote the numbers by a, b. Since adding a zero digit is the same as multiplying by ten, we only have to solve the following system of equations:

$$\begin{aligned} a + b &= 2013, \\ 10a + b &= 3858. \end{aligned}$$

Subtracting them we learn $a = 205$. Plugging it in the first one we get $b = 1808$, which is the answer.

Problem 17J / 7S. What is the radius of the smallest circle which can cover a triangle with side lengths 3, 5, and 7?

Result. 3.5

Solution outline. Since $3^2 + 5^2 < 7^2$, the angle opposite the 7-side is obtuse. Thus the triangle can be covered with a circle of radius 3.5. On the other hand, no smaller circle can cover the longest side, so 3.5 is our final answer.

Problem 18J / 8S. Lisa, Mary, Nancy and Susan have 100 lollipops altogether. Each pair of them has at least 41 lollipops. What is the minimum number of lollipops that Lisa can have?

Result. 12

Solution outline. If we denote the numbers of each girl's lollipops by L, M, N, and S, respectively, we know that $L+M \geq 41$, $L+N \geq 41$, and $L+S \geq 41$. Summing these inequalities we get $2L+(L+M+N+S) \geq 123$. Since $L+M+N+S=100$, we learn $2L \geq 23$, or $L \geq 12$ (L is a positive integer).

On the other hand, the distribution $L=12$, $M=N=29$, $S=30$ satisfies the conditions of the problem.

Problem 19J / 9S. The area of a rectangle $ABCD$ is 80 and the length of its diagonal is 16. Find the sine of the angle between its diagonals.

Result. $\frac{5}{8}=0.625$

Solution outline. Denote by S the intersection of the diagonals and by h the length of the B-altitude in $\triangle ABC$. The area of $ABCD$ can be found as $AC \cdot h$ which gives $h=80/16=5$. The sought-after value is then $\sin\angle ASB=\sin\angle CSB=\frac{h}{BS}=\frac{5}{8}$.

Problem 20J / 10S. What is the largest possible value of a^b+c^d, where a, b, c, and d are distinct elements of the set $\{-7,-5,-4,-3,-2,-1\}$?

Result. $(-1)^{-4}+(-3)^{-2}=\frac{10}{9}$

Solution outline. In order to obtain a positive number, we need to take even exponents. Also, since $(-2)^{-4}=(-4)^{-2}<(-1)^{-2}$, we will take the exponents -2 and -4. Further, as a negative power is decreasing, we will take -1 and -3 as bases. Out of the remaining two options, $(-1)^{-4}+(-3)^{-2}=\frac{10}{9}$ has greater value.

Problem 21J / 11S. An angle of magnitude $110°$ is placed in the coordinate plane at random. What is the probability that its legs form a graph of some function?

Result. $\frac{11}{18}$

Solution outline. For an angle to form a graph of a function, it must not contain two points with the same x-coordinate. We may assume that the vertex of the angle is at point $(0,0)$. We will rotate the angle by one full circle (clockwise) and determine when it is a graph of a function. We choose to start in a position in which one leg coincides with the positive ray of the y axis and the other lies to the right. Observing the breaking point after $70°$ rotation, we see that the answer is $\frac{110}{180}=\frac{11}{18}$.

Problem 22J / 12S. James drew a regular 100-gon $A_1A_2\ldots A_{100}$ (numbered clockwise) on the last year's Náboj (23rd March, 2012) and randomly placed a chip on one of the vertices. Every following day, he moved the chip clockwise by the number of vertices indicated by the number of the current vertex (from A_3 he would move it to A_6, from A_{96} to A_{92}). Now the chip lies on A_{100}. What was the probability this would happen?

Result. $0.04=\frac{1}{25}$

Solution outline. We are in fact asking about numbers from 1 to 100 which, when repeatedly multiplied by 2, become a multiple of 100. Such number must be a multiple of 25 and all four of these (25, 50, 75, 100) work. The answer is $\frac{4}{100}=0.04=\frac{1}{25}$.

Problem 23J / 13S. A positive integer is called *differential* if it can be written as a difference of squares of two integers. How many of the numbers $1,2,\ldots,2013$ are differential?

Result. 1510

Solution outline. Odd numbers can be written as $2k+1=(k+1)^2-k^2$ and multiples of four as $4k=(k+1)^2-(k-1)^2$. Numbers of the form $4k+2$ are not differential since $a^2-b^2=(a+b)(a-b)$ is a product of two positive integers of the same parity, therefore either odd or multiple of 4. There are 503 numbers of the form $4k+2$ from 1 to 2013, hence the answer is 1510.

Problem 24J / 14S. Three non-overlapping regular convex polygons with side length 1 have point A in common and their union forms a (nonconvex) polygon M which has A in its interior. If one of the regular polygons is a square and another is a hexagon, find the perimeter of M.

Result. 16

Solution outline. We look at the internal angles of the polygons by vertex A. One of them is $90°$, another is $120°$, and since the three angles add up to $360°$ (A is in the interior), the remaining angle is $150°$, which corresponds to a regular dodecagon (12 vertices). Since each pair of polygons shares a side, the perimeter of M is $4 + 6 + 12 - (3 \cdot 2) = 16$.

Problem 25J / 15S. Mike wrote the numbers from 1 to 100 in random order. What is the probability that for each $i = 1, \dots, 50$ the number on the position $2i - 1$ is smaller than the number on the position $2i$?

Result. 2^{-50}

Solution outline. Imagine Mike writes the numbers in pairs: First he randomly chooses two available numbers and then their order. The smaller of them going first thus has probability $\frac{1}{2}$ (regardless of the chosen numbers) and thus for 50 pairs the probability is $\left(\frac{1}{2}\right)^{50} = 2^{-50}$.

Problem 26J / 16S. Pink paint is made up of red and white paint in the ratio $1 : 1$, cyan paint is made up of blue and white in the ratio $1 : 2$. Karen intends to paint her room with paint which is made up of pink and cyan in the ratio $2 : 1$. She has already mixed three tins of blue and one tin of red paint. How many tins does she have to add to the mixture, provided there are only red and white tins left?

Result. 23

Solution outline. Since we can add only red or white paint, the resulting paint has to contain exactly three tins of blue. Blue is not contained in pink paint, thus the ratio $1 : 2$ in cyan implies that there need to be exactly 9 tins of cyan paint (3 blue and 6 white). Finally, from the ratio $2 : 1$ in the desired paint we infer that it has to be mixed from 27 tins (9 of which are mixed to produce cyan and 18 to pink). As we have already mixed four tins, we have to add $27 - 4 = 23$ tins to complete the paint.

Problem 27J / 17S. We have a hat with several white, gray, and black rabbits. When a magician starts taking them randomly out of the hat (without returning them), the probability of him taking a white rabbit before a gray one is $\frac{3}{4}$. The probability of him taking a gray rabbit before a black one is $\frac{3}{4}$ as well. What is the probability of him taking a white rabbit before a black one?

Result. $\frac{9}{10}$

Solution outline. The magician takes a white rabbit before a gray one with the probability of $\frac{3}{4}$, hence the hat contains three times as many white rabbits as the gray ones. In like manner, there are three times as many gray rabbits as the black ones. Therefore the hat is inhabited by nine times as many white rabbits as the black ones and the sought-after probability is $\frac{9}{10}$.

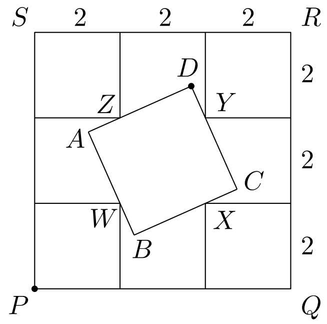
Problem 28J / 18S. Positive integers a, b satisfy $49a + 99b = 2013$. Find the value of $a + b$.

Result. 37

Solution outline. We add $a + b$ to both sides of the equation and get $50(a + 2b) = 2013 + (a + b)$. The left-hand side of the equation is divisible by 50, and so must be the right-hand side. It follows that $a + b = 50k - 13$ for some $k \in \mathbb{N}$.

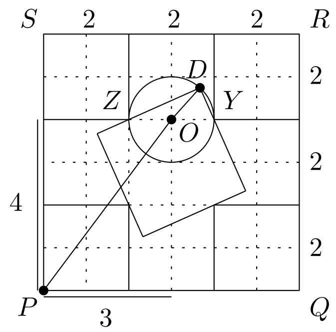
If $a + b \geq 87$, then $49a + 99b > 49a + 49b \geq 49 \cdot 87 > 2013$, which is not possible. Thus the only possibility is $a + b = 37$.

Problem 29J / 19S. In the corners of a square $PQRS$ with side length 6 cm four smaller squares are placed with side lengths 2 cm. Let us denote their vertices W, X, Y, Z like in the picture. A square $ABCD$ is constructed in such a way, that points W, X, Y, Z lie inside the sides AB, BC, CD, DA respectively. Find the largest possible distance between points P and D.



Result. 6

Solution outline. Point D belongs to the circle with diameter ZY, the center of which we denote by O. This circle has radius 1 and by Pythagorean Theorem we have $PO = \sqrt{3^2 + 4^2} = 5$. Hence from triangle inequality we obtain $PD \leq PO + OD = 6$. The equality is attained if P, O, D lie on one line, so the maximum distance is 6.



Problem 30J / 20S. We have twenty boxes of apples that altogether contain 129 apples. We know that in some of the boxes there are exactly 4 apples and all the remaining boxes contain exactly x apples. Find all possible values of x.

Result. 11, 53

Solution outline. Denote by K the number of boxes that contain x apples ($K \leq 20$). In the remaining $20 - K$ boxes we have four apples and hence

$$K \cdot x + (20 - K) \cdot 4 = 129, \quad \text{which implies} \quad K(x - 4) = 49.$$

We have $K \leq 20$, hence the only possible choices are $K = 1$ and $K = 7$. From $K = 1$ we easily obtain $x = 53$, and $K = 7$ corresponds to $x = 11$.

Problem 31J / 21S. Consider real numbers a, b such that $a > b > 0$ and

$$\frac{a^2 + b^2}{ab} = 2013.$$

Find the value of the expression $\frac{a+b}{a-b}$.

Result. $\sqrt{\frac{2015}{2011}}$

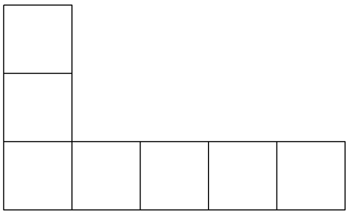
Solution outline. From the given equality we obtain $a^2 + b^2 = 2013ab$. Using this identity we have

$$(a+b)^2 = a^2 + 2ab + b^2 = 2015ab,$$
$$(a-b)^2 = a^2 - 2ab + b^2 = 2011ab.$$

From $a > b > 0$ we know that the expression $\frac{a+b}{a-b}$ is positive and hence using the two relations above yields

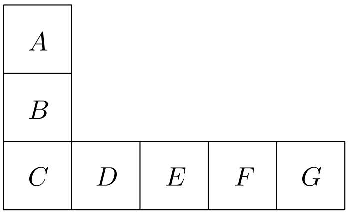
$$\frac{a+b}{a-b} = \sqrt{\frac{(a+b)^2}{(a-b)^2}} = \sqrt{\frac{2015ab}{2011ab}} = \sqrt{\frac{2015}{2011}}.$$

Problem 32J / 22S. In how many ways can we fill in individual squares in the pictured heptomino piece with numbers 1 to 7 (each of which should be used once only) so that the sum of numbers in the bottom row is the same as the sum of numbers in the left column?



Result. $144 = 3 \cdot 2 \cdot 4!$

Solution outline. Denote values in the individual squares as in the picture.



It follows from the problem statement that $A+B+C+D+E+F+G = 1+2+3+4+5+6+7 = 28$ and $A+B = D+E+F+G = \frac{1}{2}(28-C)$. Therefore C must be even.

If $C = 2$, then $A + B = 13 = 6 + 7$, which gives two alternatives how to fill in the squares in the left column. For each of them we can reorder the numbers in the bottom row in 4! ways. If $C = 4$, then $A + B = 12 = 5 + 7$ ($6 + 6$ is not legitimate since identical numbers are disallowed), and if $C = 6$, then $A + B = 11 = 4 + 7$ ($5 + 6$ is also illegitimate since $C = 6$ already), which gives in both cases $2 \cdot 4!$ possibilities of finishing the table. Hence we obtain altogether $3 \cdot 2 \cdot 4! = 144$ possibilities.

Problem 33J / 23S. Let ABC be an acute triangle with $AB = 4\pi$, $BC = 4\pi + 3$, $CA = 4\pi + 6$. Denote by D the foot of the A-altitude. Find $CD - BD$.

Result. 12

Solution outline. Pythagorean Theorem for right triangles ADC and ADB gives $CD^2 = AC^2 - AD^2$ and $BD^2 = AB^2 - AD^2$. After subtracting, we obtain

$$CD^2 - BD^2 = AC^2 - AB^2 = (4\pi + 6)^2 - (4\pi)^2 = 48\pi + 36.$$

Since D lies on the segment BC, we also have

$$CD^2 - BD^2 = (CD - BD) \cdot (CD + BD) = (CD - BD) \cdot (4\pi + 3),$$

hence $CD - BD = 12$.

Problem 34J / 24S. Trams have the same intervals all day and in both directions. A pedestrian walked along the tram rails and observed that every 12 minutes he was overtaken by a tram, and every 4 minutes one tram coming in the opposite direction passed him. What is the time interval between two trams?

Result. 6 minutes

Solution outline. We denote the speed of the tram by t, the speed of the pedestrian by c and the distance between two trams by d. From the given information we deduce that

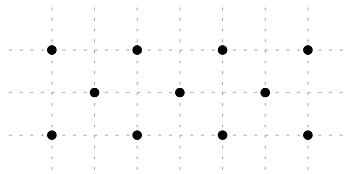
$$\begin{aligned} t-c &= \frac{d}{12\,\mathrm{min}}, \\ t+c &= \frac{d}{4\,\mathrm{min}}. \end{aligned}$$

By summing up the equations and dividing by 2 we obtain

$$v = \frac{1}{2}\left(\frac{1}{12\,\mathrm{min}} + \frac{1}{4\,\mathrm{min}}\right)\cdot d = \frac{d}{6\,\mathrm{min}}.$$

It follows that the tram travels the distance d in 6 minutes, which implies that the interval between two trams is 6 minutes.

Problem 35J / 25S. How many nondegenerate triangles can be formed by means of joining some point triplet from the picture?



Note: Points are aligned in the indicated grid.

Result. $148 = \binom{11}{3} - 17$

Solution outline. We can pick a point triplet out of the given eleven ones in

$$\binom{11}{3} = \frac{11\cdot 10\cdot 9}{1\cdot 2\cdot 3} = 165$$

ways. It remains to compute the number of point triplets lying on a line and to subtract the resulting number from the one above. There are $4+1+4=9$ horizontal triplets, $3+3+1+1=8$ transversal triplets, hence the number of triangles is $165-17=148$.

Problem 36J / 26S. Thomas got a box of chocolates with 30 sweets arranged in three rows by ten. He is eating the sweets one by one in such a way that the number of sweets in two rows always differs at most by one. In how many ways can he eat the whole box?

Result. $6^{10}\cdot(10!)^3$

Solution outline. The way of eating the sweets can be uniquely determined in the following way: For each row we choose the order of its sweets in which Thomas will eat them and also choose the order of the rows in which he will switch between them.

For each row, we have $10!$ possibilities to choose the order of sweets, which gives us $(10!)^3$ possibilities for three rows.

Now we need to count the number of the order of rows. We know that if there is the same number of sweets in all rows, then Thomas can choose any of them, then he needs to choose one of the remaining two (choosing the same row would result in having two sweets less in this row) and then he has to choose the remaining one. After three steps he has again the same number of sweets in all three rows. It follows that it is enough to choose ten times the order of the three rows, which can be done in $(3!)^{10} = 6^{10}$ different ways.

Thomas can eat the box of chocolates in $6^{10}\cdot(10!)^3$ different ways.

Problem 37J / 27S. We call a positive integer with six digits *doubling* if the first three digits are the same (including the order) as the other three digits. For example the number 227227 is doubling, but 135153 is not doubling. How many six-digit doubling numbers are divisible by 2013?

Note: The first digit of a positive integer cannot be 0.

Result. 5

Solution outline. Any doubling number with six digits can be written as $1001 \cdot k$ for some positive integer k. Conversely, for any three digit positive integer k the formula $1001 \cdot k$ gives us a six-digit doubling number.

As $2013 = 3 \cdot 11 \cdot 61$ and 1001 is divisible by 11, but not by 3 or 61, it follows that the six digit integer is doubling if and only if the corresponding three digit k in $1001 \cdot k$ is a multiple of $3 \cdot 61 = 183$. We have exactly five three digit k divisible by 183 and hence we have five six-digit doubling numbers that are divisible by 2013.

Problem 38J / 28S. Suppose that we have 4×4 chessboard and on each square we draw an arrow aiming right or aiming down at random. We put a robot on the upper left corner and let it move along the chessboard in the directions of the arrows. What is the probability that it will exit the chessboard from the square in the lower right corner?

Result. $\frac{5}{16} = \frac{\binom{6}{3}}{2^6}$

Solution outline. Let us count the number of paths from the upper left to the lower right corner and the probability that the robot will go along one such path. Each path consists of three steps down and three steps right which gives us $\binom{6}{3}$ possible paths. The probability that the robot will follow this path is always 2^{-6} as in each of the steps he must choose the correct direction. The overall probability is thus $2^{-6} \cdot \binom{6}{3}$.

Problem 39J / 29S. Write

$$\frac{212121210}{112121211}$$

in the basic form (i.e. as the fraction $\frac{a}{b}$, where a and b are positive integers with no common divisor).

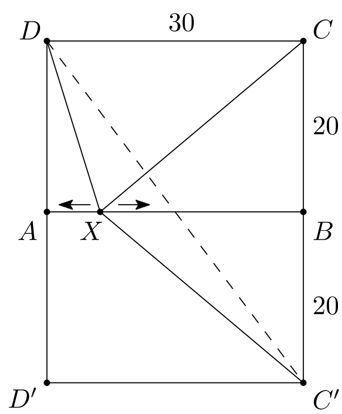
Result. $\frac{70}{37}$

Solution outline. It is easy to see (as the sum of the digits is divisible by three) that both numerator and denominator are divisible by three. By division we get $212121210 = 3 \cdot 70707070$ and $112121211 = 3 \cdot 37373737$. Now observe that $70707070 = 70 \cdot 1010101$ and $37373737 = 37 \cdot 1010101$, which implies that the basic form of our fraction is $\frac{70}{37}$.

Problem 40J / 30S. Let $ABCD$ be a rectangle with side lengths $AB = 30$ and $BC = 20$. For how many points X on its side AB is the perimeter of the triangle CDX an integer?

Result. 13

Solution outline. It is sufficient to find when is $DX + XC$ an integer. Let us consider C', D' such that A, B are midpoints of DD', CC' respectively. Then $DX + XC = DX + XC'$. The right-hand side attains its minimum when X is the midpoint of AB and maximum when X is equal to A or B.



Using Pythagorean Theorem we compute $DC' = \sqrt{30^2+40^2} = 50$ and $AC' = \sqrt{30^2+20^2} = \sqrt{1300}$, from which we have $36 < AC' < 37$. So, if X moves from A to B, then $DX + XC$ first decreases from a number a little larger than $20+36 = 56$ ($X = A$) to 50 (X is the midpoint of AB) and then increases to a number a little larger than 56 ($X = B$). Hence, it is integer for $6+1+6 = 13$ positions of the point X.

Problem 41J / 31S. In what order should we place the rows $r_1, \ldots, r_{11}$ of the table in the picture, if we want the newly created table to be symmetric with respect to the marked diagonal? It is sufficient to find one solution.

r_1	1	0	0	1	1	0	0	0	1	0	0
r_2	0	1	0	0	1	1	0	0	0	1	0
r_3	0	0	1	0	0	1	1	0	0	0	1
r_4	1	0	0	1	0	0	1	1	0	0	0
r_5	0	1	0	0	1	0	0	1	1	0	0
r_6	0	0	1	0	0	1	0	0	1	1	0
r_7	0	0	0	1	0	0	1	0	0	1	1
r_8	1	0	0	0	1	0	0	1	0	0	1
r_9	1	1	0	0	0	1	0	0	1	0	0
r_{10}	0	1	1	0	0	0	1	0	0	1	0
r_{11}	0	0	1	1	0	0	0	1	0	0	1

Result. in the reverse order (eventually complemented by a circular shift), i.e. $r_{11}, r_{10}, r_9, \ldots, r_1$ or $r_{10}, r_9, \ldots, r_1, r_{11}$, …, $r_1, r_{11}, r_{10}, \ldots, r_2$

Solution outline. Observe that the table is symmetric with respect to the other diagonal. To make it symmetric with respect to the marked diagonal it is sufficient to reflect it over the horizontal axis.

Remark: For this table there are no solutions except the eleven mentioned above.

Problem 42J / 32S. For every positive integer n let

$$a_n = \frac{1}{n}\sqrt[3]{n^3+n^2-n-1}.$$

Find the smallest integer $k \geq 2$ such that $a_2 \cdot a_3 \cdots a_k > 4$.

Result. 254

Solution outline. Denoting $A_n = a_n^3$ our problem is equivalent to finding the smallest integer $k \geq 2$ such that $A_2 \cdot A_3 \cdots A_k > 4^3 = 64$. We have

$$A_n = \frac{n^3+n^2-n-1}{n^3} = \frac{(n+1)\cdot(n+1)\cdot(n-1)}{n\cdot n\cdot n}$$

and hence

$$A_2 \cdot A_3 \cdots A_k = \frac{3\cdot 3\cdot 1}{2\cdot 2\cdot 2}\cdot\frac{4\cdot 4\cdot 2}{3\cdot 3\cdot 3}\cdots\frac{(k+1)\cdot(k+1)\cdot(k-1)}{k\cdot k\cdot k} = \frac{1\cdot(k+1)\cdot(k+1)}{2\cdot 2\cdot k} = \frac{(k+1)^2}{4k}.$$

It remains to solve the inequality

$$(k+1)^2 > 256k$$

for integer k. Substracting $2k$ from both sides and multiplying them by $1/k$ yields equivalent inequality $k + \frac{1}{k} > 254$, the smallest integer solution of which is $k = 254$.

Problem 43J / 33S. Barbara cut the pizza into n equal slices and then she labeled them with numbers $1, 2, \ldots, n$ (she used each number exactly once). The numbering had the property that between each two slices with consecutive numbers (i and $i+1$) there was always the same number of other slices. Then came the fatty Paul and ate almost the whole pizza, only the three neighboring slices with numbers 11, 4, and 17 (in this exact order) remained. How many slices did the pizza have?

Result. 20

Solution outline. Suppose that between the slices with consecutive numbers we have exactly $k-1$ other slices, that is by moving by k slices we get from slice with number 1 to slice with number 2, from slice 2 to slice 3 and so on. All these movements have to be in the same direction, otherwise we would get from slice i to the previous slice $i-1$ and not to $i+1$. From the slice n we get in this way necessarily to slice number 1, because any other i has in distance k slices with numbers $i+1$ and $i-1$. By moving with the step k slices we will finally pass through the whole pizza. Hence there is s such that moving s times by k slices we will end exactly at the neighboring slice. Thus we get

$$11-4 \equiv s \cdot k \equiv 4-17 \pmod{n},$$

which implies that $7-(-13)=20$ is divisible by n. There is a slice with number 17 and hence $n \geq 17$ which implies that the only solution is $n=20$.

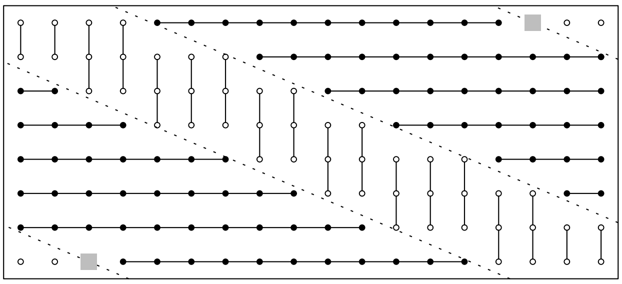
Problem 44J / 34S. In one of the lecture halls at Matfyz the seats are arranged in a rectangular grid. During the lecture of analysis there were exactly 11 boys in each row and exactly 3 girls in each column. Moreover, two seats were empty. What is the smallest possible number of the seats in the lecture hall?

Result. 144

Solution outline. We denote by r and s the number of rows and columns in the lecture hall. From the instructions we have $rs = 11r + 3s + 2$, which is equivalent to

$$(r-3)(s-11) = 35.$$

The numbers in the brackets are either 5 and 7, or 1 and 35 in some order. By writing down all four possibilities we can find that the smallest number rs corresponds to the case $r-3=5$, $s-11=7$, when $rs = 8 \cdot 18 = 144$. It remains to show that we can really put the students in the lecture hall as requested, which can be seen in the picture.



Problem 45J / 35S. Circle k with radius 3 is internally tangent to circle l with radius 4 at point T. Find the greatest possible area of triangle TKL, where $K \in k$ and $L \in l$.

Result. $9\sqrt{3} = \frac{27}{\sqrt{3}}$

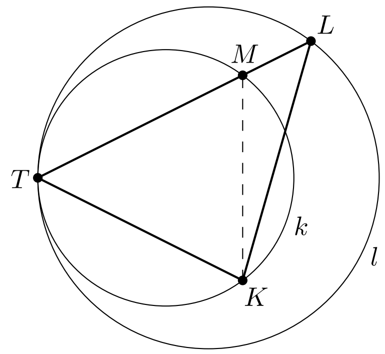
Solution outline. Denote by $[XYZ]$ the area of triangle XYZ.

Denote by M the intersection of TL with k ($M \neq T$). Since T is the center of homothety with factor $\frac{4}{3}$ sending k to l, points L and M correspond and thus $TL = \frac{4}{3}TM$ and $[TKL] = \frac{4}{3}[TKM]$ (the triangles share an altitude from K). Hence it suffices to maximize the area of TKM inscribed in circle k with radius 3. From all such triangles the equilateral one has the greatest area, namely

$$3 \cdot \left(\frac{1}{2} \cdot 3 \cdot 3 \sin 120^\circ\right) = \frac{27\sqrt{3}}{4}.$$

Hence the area of the corresponding triangle TKL is

$$\frac{4}{3} \cdot \frac{27\sqrt{3}}{4} = 9\sqrt{3}.$$



Problem 46J / 36S. Alice and Bob are playing the following game: At the beginning they have a set of numbers $\{0, 1, \ldots, 1024\}$. First Alice removes any 2^9 elements, then Bob removes any of the remaining 2^8 elements, then Alice removes 2^7 elements and so on until at last Bob removes one element and there are exactly two numbers remaining. The game ends then and Alice pays to Bob the absolute value of the difference of these two numbers in Czech crowns. How many crowns will Bob get if they both play in the best way they can?

Result. 32

Solution outline. Bob can double the smallest distance between any two numbers in each step by removing every other element. In this way he wins at least $2^5 = 32$ crowns. Alice can halve the difference of the largest and the smallest number in each step by removing upper half (or lower half) of all remaining numbers. This ensures that she loses at most $1024/2^5 = 32$ crowns. Therefore if they both play in the best way, Bob will get 32 crowns.

Problem 47J / 37S. Some students of Matfyz did not pass the exams and were expelled. They all started to study at SOU (some other university). It had the following consequences:

1. The number of students of Matfyz decreased by one sixth.
2. The number of students of SOU increased by one third.
3. The average IQ on both schools increased by 2%.

How many times was the average IQ at Matfyz higher than average IQ at SOU?

Result. $\frac{6}{5} = 1.2$-times

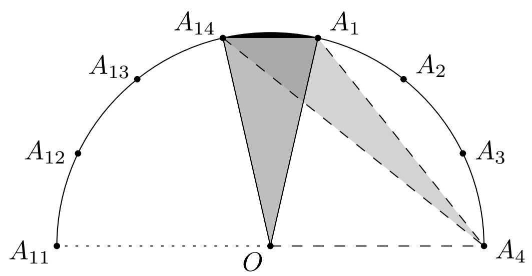
Solution outline. We denote by $100m$ the original average IQ of students at Matfyz and $100v$ the original average IQ of students at SOU. Further, we denote by p the average IQ of students who changed from Matfyz to SOU. The average IQ at Matfyz increased by 2% and thus the average IQ of remaining students at Matfyz is $102m$. From the ratio $5 : 1$ of remaining students and from the original average $100m$ we obtain the equation $100m = \frac{5}{6} \cdot 102m + \frac{1}{6}p$, which is the same as $p = 90m$. Analogously the average IQ at SOU was originally $100v$ and with the new students it has risen to $102v$. From the ratio $3 : 1$ we have the equation $102v = \frac{3}{4} \cdot 100v + \frac{1}{4}p$, which implies $p = 108v$. Altogether this gives us

$$90m = 108v, \quad \text{and hence} \quad \frac{m}{v} = \frac{108}{90} = \frac{6}{5}.$$

Problem 48J / 38S. A regular tetradecagon $A_1A_2 \ldots A_{14}$ is inscribed in a circle k of radius 1. How large is the part of the disc circumscribed by the circle k, which lies inside the angle $\angle A_1A_4A_{14}$?

Result. $\frac{\pi}{14}$

Solution outline. Let us focus on points A_1, A_4, A_{14} and A_{11}.



Since $11 = 4 + 7$, the line segment A_4A_{11} is a diameter of the circle k. On the other hand we have $4-1 = 3 = 14-11$, so $A_1A_4A_{11}A_{14}$ is an isosceles trapezoid and its bases A_4A_{11} and A_1A_{14} are parallel. The area of the triangle $A_1A_4A_{14}$ is therefore the same as the area of the triangle A_1OA_{14}, where O is the center of k. The area to compute is then equal to the area of the sector A_1OA_{14}, i.e. one fourteenth of the area of the disc.

Problem 49J / 39S. Bill and Carol saw the 24-element set $\{1, 2, \ldots, 24\}$. Bill wrote down all its 12-element subsets whose sum of the elements was even. Carol wrote down all its 12-element subsets whose sum of the elements was odd. Who wrote more subsets and by how many?

Result. Bill, $\binom{12}{6} = 924$ subsets more

Solution outline. Consider an arbitrary 12-element subset B and assume there exists an integer i such that B contains exactly one of the numbers $2i-1$, $2i$. Let us take the smallest such i and construct a 12-element set $f(B)$, which contains the same elements as B with the only exception: It contains the other element from the couple $2i-1$, $2i$.

It is easy to see that $f(f(B)) = B$ and that by applying f on a Bill's subset we obtain Carol's subset and vice versa. So, f is a bijection between Bill's and Carol's subsets, if we consider just those subsets for which there exists an i from the previous paragraph. It remains to count the remaining subsets, for which such i does not exist.

In each of the remaining subsets, there must be exactly six odd numbers and six even numbers which are successors of the odd ones. Sum of the elements of such subsets is always even (hence they belong to Bill's list) and the number of such subsets is $\binom{12}{6}$.

Problem 50J / 40S. Jack thought of three distinct positive integers a, b, c such that the sum of two of them was 800. When he wrote numbers a, b, c, $a+b-c$, $a+c-b$, $b+c-a$ and $a+b+c$ on a sheet of paper, he realized that all of them were primes. Determine the difference between the largest and the smallest number on Jack's paper.

Result. 1594

Solution outline. Let us assume without loss of generality that $b + c = 800$. At least one of the numbers a, $b+c-a = 800-a$, $a+b+c = 800+a$ is divisible by three, so it can be prime number only if it is equal to three. Since $800 + a > 800$, there are only two possibilities, $a = 3$ or $800 - a = 3$.

If $a = 3$, then we have $3 + (b - c) \geq 2$ and simultaneously $3 - (b - c) \geq 2$, so $|b - c| \leq 1$. This is impossible, since $b + c = 800$.

So, we know that $800 - a = 3$, i.e. $a = 797$. The largest among Jack's prime numbers is $a + b + c = 797 + 800 = 1597$. Since $b + c = 800$, none of the Jack's primes is even. Therefore, the smallest number is $800 - a = 3$. The difference is then $1597 - 3 = 1594$.

Let us remark that such numbers exist, we can take $a = 797$, $b = 223$, $c = 577$.

Problem 51J / 41S. Helen stained two randomly chosen places on a one-meter bar. Thereupon Alex came and shattered the bar, just as well randomly, into 2013 pieces. What is the probability of both stains being now on the same piece?

Result. $\frac{1}{1007}$

Solution outline. Imagine the bar is in one piece. Helen marked it randomly with two stains, whereas Alex marked it randomly with 2012 breaks. The bar thus contains altogether 2014 marks, two of which are stains. The total number of possibilities of which marks can be stains is $\binom{2014}{2} = 1007 \cdot 2013$. The stains are on the same piece (or shard) if and only if they are neighbouring marks, which occurs in 2013 cases. The resultant probability is

$$\frac{2013}{1007 \cdot 2013} = \frac{1}{1007}.$$

Problem 52J / 42S. How many ten-digit positive integers that are made up of $0, 1, \ldots, 9$ (i.e. each digit is used exactly once) are multiples of 11111?

Note: The first digit of a positive integer cannot be 0.

Result. $3456 = 2^5 \cdot 5! - 2^4 \cdot 4!$

Solution outline. Since $0+1+\cdots+9 = 9\cdot 5$, the numbers under investigation must be divisible by nine, hence even by 99999. Let us denote A and B the numbers consisting of the former and the latter five-tuple of the investigated number respectively. We have

$$99999 \mid 100000A + B, \text{ iff } 99999 \mid A + B.$$

Given that A, B are five-digit positive integers less than 99999, we observe

$$0 < A + B < 2\cdot 99999, \text{ hence } A + B = 99999, \text{ or equally } B = 99999 - A.$$

Therefrom we obtain the necessary and sufficient condition on A, B for divisibility of the investigated ten-digit number by 99999: For $i = 1,\ldots,5$, the i-th digit of B is a complement of i-th digit of A into nine. We couple the available digits into five pairs

$$(0,9),(1,8),(2,7),(3,6),(4,5).$$

We know that these pairs must be used in certain order (5! options) and at the same time, in each pair we may choose, which digit will be put in A and which in B (2^5 options). However, we have to remove the choices including a zero as the first digit in A – that is 4! options of redistributing the remaining pairs and 2^4 options of redistributing their digits between A and B. The amount of desired numbers is thus $5!\cdot 2^5 - 4!\cdot 2^4$.

Problem 53J / 43S. A polynomial $P(x)$ of degree 2013 with real coefficients fulfills for $n = 0,1,\ldots,2013$ the relation $P(n) = 3^n$. Evaluate $P(2014)$.

Result. $3^{2014} - 2^{2014}$

Solution outline. Define the polynomial $Q(x) = \sum_{k=0}^{2013}\binom{x}{k}2^k$. Its degree is 2013 and moreover, for any $x \in \{0,\ldots,2013\}$ the binomial theorem yields

$$Q(x) = \sum_{k=0}^{2013}\binom{x}{k}2^k = \sum_{k=0}^{x}\binom{x}{k}2^k = (1+2)^x = P(x).$$

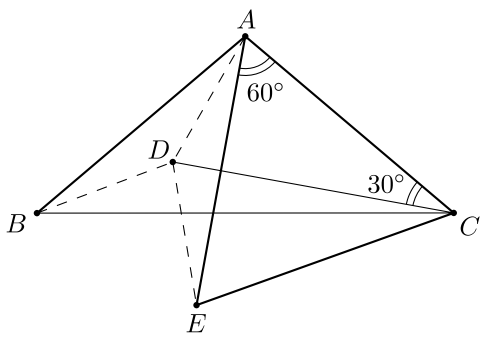
The polynomial $P(x) - Q(x)$ is of degree 2013 and it possesses 2014 roots, whereby it must be zero. Hence $P(x) = Q(x)$ and it only remains to compute

$$Q(2014) = \sum_{k=0}^{2013}\binom{2014}{k}2^k = \sum_{k=0}^{2014}\binom{2014}{k}2^k - \binom{2014}{2014}2^{2014} = (1+2)^{2014} - 2^{2014} = 3^{2014} - 2^{2014}.$$

Problem 54J / 44S. Inside an isosceles triangle ABC fulfilling $AB = AC$ and $\angle BAC = 99.4^\circ$, a point D is given such that $AD = DB$ and $\angle BAD = 19.7^\circ$. Compute $\angle BDC$.

Result. 149.1°

Solution outline. Denote by E the image of B in reflection over AD.



Then $AE = AB = AC$ and $\angle EAC = \angle BAC - 2\cdot\angle BAD = 60^\circ$, entailing that the triangle AEC is equilateral and $CE = CA$. In addition, $DE = DB = DA$ due to symmetry, and therefore CD is the perpendicular bisector of AE and $\angle ACD = \frac{1}{2}\angle ACE = 30^\circ$. Now it is simple to use the nonconvex quadrilateral $ABDC$ for computing

$$\angle BDC = \angle DBA + \angle BAC + \angle ACD = 19.7^\circ + 99.4^\circ + 30^\circ = 149.1^\circ.$$

Problem 55J / 45S. Find the largest positive integer not ending with a zero such that removal of one of its "inner" digits produces its divisor.

Note: An "inner" digit denotes any digit except for the first and the last one.

Result. 180625

Solution outline. Let X be the sought-after number. First we deduce the digit to be removed must be the second one.

Proceeding by contradiction, assume the two initial digits have remained. Out of an n-digit number X, the removal produces an $(n-1)$-digit number X'. The number $10 \cdot X'$ has then n digits again, the two initial ones being the same as in X, yet $X \neq X'$ because the former does not end with a zero. This is a contradiction – subtracting two multiples of an $(n-1)$-digit number cannot produce an $(n-2)$-digit number.

Now we write $X = a \cdot 10^{n+1} + b \cdot 10^n + c$, where a and b are digits ($a \neq 0$) and $c < 10^n$ is a number with a nonzero terminal digit. Removal of the second digit produces $X' = a \cdot 10^n + c$. Hence, for a suitable $k \in \mathbb{N}$, it follows that

$$a \cdot 10^{n+1} + b \cdot 10^n + c = k \cdot (a \cdot 10^n + c).$$

At this point we observe $k < 20$. Indeed, provided $k \geq 20$, the number X would have a greater initial digit than X', which is impossible. Let us modify the equality into

$$10^n(10a + b - k \cdot a) = (k-1)c.$$

Given that the left-hand side is divisible both by 2^n and 5^n, the right-hand side must share the same property. The number c does not end with a zero, hence $k-1$ has to be divisible by at least one of the prime numbers 2, 5 in their full power. Seeing that $k < 20$, we conclude $n \leq 4$ (by reason of $2^5 > 20$ and even $5^2 > 20$), resulting in X having at most 6 digits. On the other hand, for $n = 4$ it must be $k - 1 = 16$, yielding

$$5^4(b - 7a) = c.$$

In order for the right-hand side to be nonnegative, the only option is $a = 1$ (a and b are digits). As for b, we may choose $b = 8$, $b = 9$, the latter of which we disregard for c would end with a zero. With $b = 8$, we calculate $c = 5^4 = 625$ and finally verify that $X = 1 \cdot 10^5 + 8 \cdot 10^4 + 625 = 180625$ really is a solution to the problem.

Problem 56J / 46S. Three mutually distinct real numbers a, b, c satisfy

$$a = (b-2)c, \quad b = (c-2)a, \quad c = (a-2)b.$$

Compute the product abc.

Result. 3

Solution outline. If any of a, b, c is zero, then all the numbers are zero, which contradicts their mutual distinctness. Similarly, the numbers a, b, c cannot be three.

We use the third equality to express c in the first and the second equation and then we modify the second expression $b = (ab - 2b - 2)a$ into $b(a^2 - 2a - 1) = 2a$. Since the right-hand side is nonzero, the same must hold also on the left; it is therefore legitimate to divide by $a^2 - 2a - 1$ and hence to express b by means of a. We insert the result into the first expression. After some modifications we obtain

$$a(a-3)(a^3 + 3a^2 - 9a - 3) = 0,$$

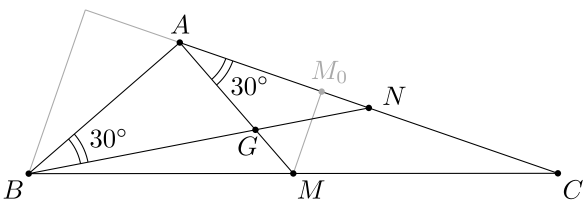
hence a is a root of the polynomial $P(x) = x^3 + 3x^2 - 9x - 3$. By analogy, we derive that b and c are roots of the same polynomial as well. Since a, b, c are mutually distinct, we infer $P(x) = (x-a)(x-b)(x-c)$. Comparing coefficients of the absolute term, we get $abc = 3$.

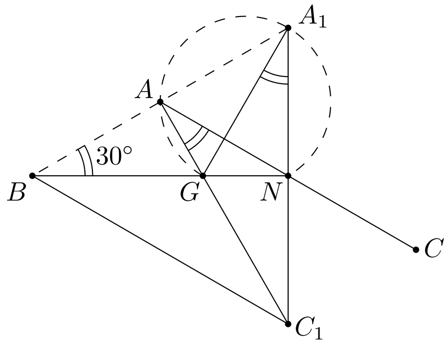
Problem 57J / 47S. In a scalene triangle ABC, there is an altitude of the same length as one median and another altitude of the same length as another median. In what ratio are the lengths of the third altitude and the third median?

Result. $\frac{2}{7}$

Solution outline. Without loss of generality assume $a > b > c$. The corresponding altitudes and medians then satisfy $v_a < v_b < v_c$ and $t_a < t_b < t_c$ respectively. At the same time, $v_a < t_a$, $v_b < t_b$ and $v_c < t_c$, which implies $v_b = t_a$ and $v_c = t_b$.

Let us denote by M the midpoint of the side BC and by M_0 the projection of M to the side AC. In the right triangle AMM_0 we observe $MM_0 = \frac{1}{2}v_b = \frac{1}{2}t_a = \frac{1}{2}AM$, hence $\angle MAC = 30°$. Denoting by N the midpoint of the side AC, we obtain similarly $\angle NBA = 30°$.





Now let G be the centroid of triangle ABC. Consider the equilateral triangle A_1BC_1 with BN being one of its medians. The point A_1 fulfills $\angle NBA_1 = 30°$ as well as $\angle GA_1N = 30°$ and yet it is distinct from the point A (the triangle ABC is scalene). "The real" point A must be therefore the other intersection of the ray BA_1 with the arc GA_1N, that is, the midpoint of the line segment BA_1. Consequently $\angle BAC = 120°$ and $AC : AB = 2$.

In a triangle with an angle $\alpha = 120°$ and side lengths $AB = 1$, $AC = 2$ we employ the Law of Cosines to calculate $a = \sqrt{1^2 + 1 \cdot 2 + 2^2} = \sqrt{7}$ and $t_c = \sqrt{1/4 + 1 + 4} = \frac{1}{2}\sqrt{21}$, then the area $S = \frac{1}{2} \cdot 1 \cdot 2 \cdot \sin 120° = \frac{1}{2}\sqrt{3}$ and finally $v_a = 2S/a = 3/\sqrt{21}$. Putting everything together we thus obtain

$$\frac{v_a}{t_c} = \frac{\frac{3}{\sqrt{21}}}{\frac{1}{2}\sqrt{21}} = \frac{2}{7}.$$