Problem 1J. Little Peter is a cool guy, so he wears only pairs of socks consisting of two socks of different colours. There are 30 red, 40 green, and 40 blue socks in his wardrobe in an unlighted cellar room. Peter takes one sock after the other out of the wardrobe without being able to recognize its colour. What is the minimum number of socks he has to take out if he needs to get eight two-coloured pairs of socks? Note that one single sock must not be counted in two different pairs.

Result. 48

Solution. If Peter pulls out all the green socks and seven red socks, he cannot form eight pairs as desired, so 47 is not sufficient. But if he takes out 48 socks, there are at least $\frac{48}{3} = 16$ socks of one colour and at least $48 - 40 = 8$ ones not having this colour among them, which enables him to form eight two-coloured pairs of socks.

Problem 2J. Assume that x and y are positive integers satisfying $x^2 + 2y^2 = 2468$. Find x if you know that there is only one such pair (x, y) and $1234 = 28^2 + 2 \cdot 15^2$.

Result. 30

Solution. Using the given equality $1234 = 28^2 + 2 \cdot 15^2$, we get

$$2468 = 2(28^2 + 2 \cdot 15^2) = (2 \cdot 15)^2 + 2 \cdot 28^2.$$

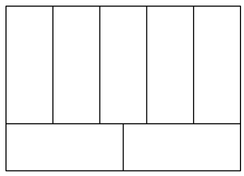
Since we know that there is only one such pair, we obtain $x = 30$.

Problem 3J. A digital watch displays time in hours and minutes using the twenty-four hour time format. How many minutes per day can one see the digit 5 on the display?

Result. 450

Solution. There are two hours during which the digit five is displayed all the time: 5 and 15. That makes 120 minutes. In the rest of the day, five can be seen during the last ten minutes of each hour $(22 \cdot 10 = 220$ minutes) and then five times during the remaining fifty minutes $(22 \cdot 5 = 110$ minutes). To sum up, five is shown exactly 450 minutes.

Problem 4J. A big rectangle with perimeter 136 cm is divided into seven congruent rectangles as in the picture.



What is the area of the big rectangle in cm^2?

Result. 1120

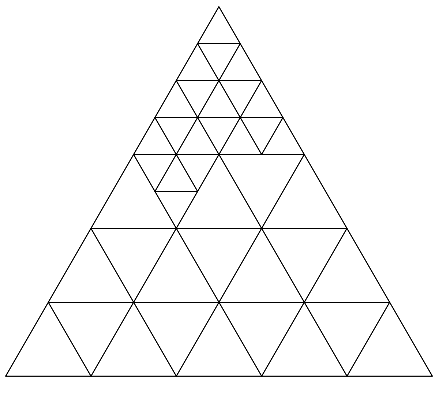
Solution. Since the side lengths of the small rectangles are in ratio $2 : 5$, let us denote them by $2x$ and $5x$. The side lengths of the big rectangle are $10x$ and $7x$, hence $2 \cdot (10x + 7x) = 34x$ is the perimeter of the big rectangle. This yields $x = 4$ cm, thus the area is $10 \cdot 7 \cdot 4^2$ cm$^2 = 1120$ cm^2.

Problem 5J. A box of chocolates has the shape of an equilateral triangle of side length s cm. There are $2n$ chocolates in the shape of equilateral triangles, which tightly fill the box: n of side length 1 cm and n of side length 2 cm. What is the smallest possible value of s?

Result. 10

Solution. Let a be the area of a small chocolate, i.e. the one with side length 1 cm. Then the area of a big chocolate is $4a$, the total area of all chocolates is $na + 4na = 5na$, and the area of the box is s^2a as the shape of the box is just the shape of a small chocolate stretched by factor s in all directions. It follows that $5n = s^2$, so s is a multiple of 5.

It can be proven that it is impossible to fit five big chocolates in a box of side length 5 cm, so $s \neq 5$. However, one can easily find a composition of 20 small and 20 big chocolates inside a box of side length 10 cm.



Problem 6J. Little Peter has grown up, so now he wears pairs of socks consisting of two socks of the same colour. He has also got many new socks, so there are currently 20 brown, 30 red, 40 green, 40 blue, 30 black, and 20 white socks in his wardrobe. However, the wardrobe is still situated in an unlighted cellar room. What is the minimum number of socks Peter has to take in order to get eight pairs of socks, if he cannot recognize their colours during the procedure? Note that one single sock must not be counted in two different pairs.

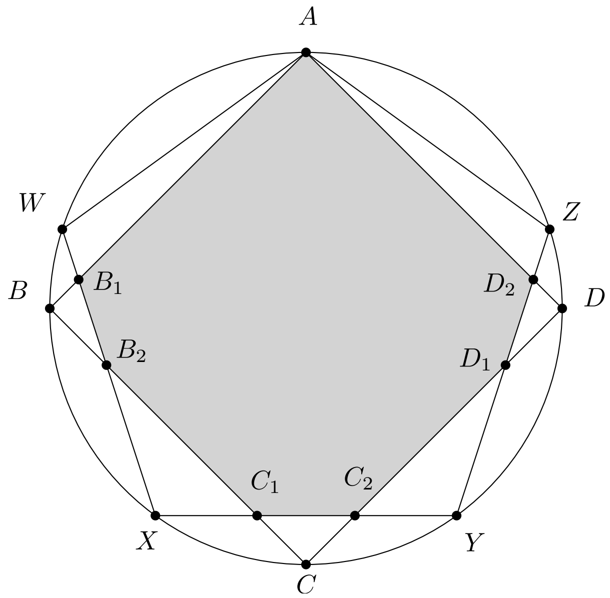
Result. 21

Solution. On one hand, every number of socks chosen is the sum of an even number of socks, which are used to build pairs of socks, and a certain number of unpaired socks, which can be at most six due to the number of colours. If Peter takes 21 socks out of his wardrobe, there cannot be six unpaired socks among them because $21 - 6 = 15$ is not even. Thus there are at most five single socks and the remaining socks form at least $\frac{21-5}{2} = 8$ pairs. On the other hand, it does not suffice to take 20 socks because Peter could have taken seven pairs of white socks and six single socks, one of each colour. It follows that Peter has to take at least 21 socks out of his wardrobe.

Problem 7J. A square and a regular pentagon are inscribed in the same circle and they share a vertex. What is the largest interior angle of the polygon which is the intersection of the two polygons?

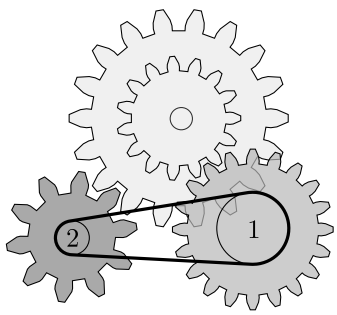
Result. 153°

Solution. Denote the vertices as in the picture. Then the polygon $AB_1B_2C_1C_2D_1D_2$ is the intersection of the two given polygons.



Since the figure is symmetric with respect to the line AC, it suffices to check the interior angles at A, B_1, B_2, and C_1. Clearly, the first one is 90° and the last one is 135°. As B_2D_1 is parallel to XY, we have $\angle D_1B_2B_1 = \angle YXW = 108°$, and since $\angle C_1B_2D_1 = \angle CBD = 45°$ ($B_2D_1 \parallel BD$), the angle at B_2 is 153°. Finally, triangle B_1BB_2 is right-angled, from which we easily obtain that the angle at B_1 is 117°. Thus the largest angle is 153°.

Problem 8J. The circle 1 has diameter 48 mm. What diameter should the circle 2 have to make the whole mechanism work?



Result. 20 mm

Solution. Having counted the numbers of teeth, one can observe that one full rotation of wheel 1 forces $\frac{20}{15} = \frac{4}{3}$ of a full circle of the double-wheel. Similarly, when the double-wheel performs one full rotation, wheel 2 covers $\frac{18}{10} = \frac{9}{5}$ of a full circle. Hence whenever wheel 1 rotates one full circle, wheel 2 rotates $\frac{4}{3} \cdot \frac{9}{5} = \frac{12}{5}$ of the full circle. The circumference of circle 2 must therefore be $\frac{5}{12}$ times the circumference of circle 1. The ratio of circumferences equals the ratio of diameters, which allows us to compute the diameter of circle 2 as $\frac{5}{12} \cdot 48$ mm $= 20$ mm.

Problem 9J. For his sixty-two-day summer holiday in July and August, Robert had prepared a precise plan on which days he would lie and on which he would tell the truth. On the k-th day of the holiday (for each k from 1 to 62), he reported that he had planned to lie on at least k days. How many of these statements were lies?

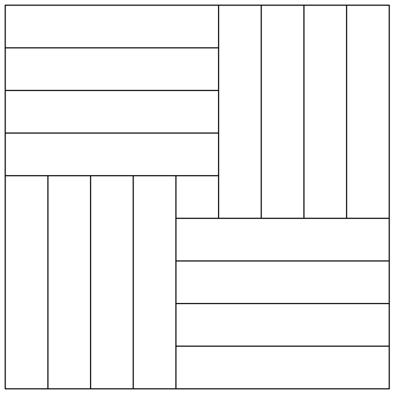
Result. 31

Solution. Observe that if Robert was telling the truth on one day, then he must have been doing so on all the previous days. If he was telling the truth only for $k < 31$ days, it would contradict the fact that he was lying for $62 - k > 31$ days. Similarly, telling the truth for more than 31 days would result in lying too little. We infer that Robert lied on exactly 31 days.

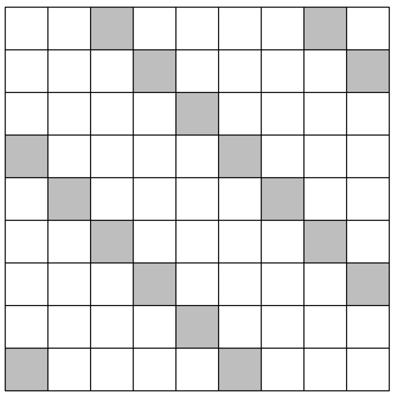
Problem 10J. In a game of battleship, our opponent has hidden an aircraft carrier, represented by a 5×1 or 1×5 block, somewhere inside a 9×9 cell grid. What is the smallest number of shots, i.e. choices of a cell in the grid, that is needed to ensure that we have hit the carrier at least once?

Result. 16

Solution. The decomposition of the 9×9 cell grid in figure 1 shows that 16 shots are necessary since every 5×1 rectangle must be hit by at least one shot.



But 16 shots are sufficient as well, as we see from figure 2.



Therefore 16 is the correct answer.

Problem 11J / 1S. What is the maximum possible value of a common divisor of distinct positive integers a, b, c satisfying $a + b + c = 2015$?

Result. 155

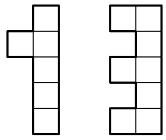
Solution. Since $5 \cdot 13 \cdot 31 = 2015 = a + b + c = \gcd(a, b, c) \cdot (a_0 + b_0 + c_0)$ for some distinct positive integers a_0, b_0, c_0, we obtain $a_0 + b_0 + c_0 \geq 6$. Hence $\gcd(a, b, c)$ can be at most $5 \cdot 31 = 155$. We can reach the maximum by taking any distinct positive integers that sum up to 13. For instance, $a_0 = 1$, $b_0 = 5$, $c_0 = 7$ yield $a = 1 \cdot 155$, $b = 5 \cdot 155$, $c = 7 \cdot 155$.

Problem 12J / 2S. A train supplying an ironworks consists of a locomotive (which is always in the front) and six carriages, each carrying either coal or iron ore. Adam wanted to take a picture of the train, but he failed to capture the whole train, so only an iron ore carriage directly followed by two coal carriages was visible in the photo. The carriages were not completely symmetric, so it was clear that the ore carriage was the first one of these three. How many different trains can be photographed so that one obtains the same picture as Adam did?

Result. 31

Solution. There are four possibilities where the photographed I-C-C carriage sequence can be, and for each of them there are 2^3 ways to complete the train. However, this results in counting the train I-C-C-I-C-C twice. Thus the number of different trains is $4 \cdot 8 - 1 = 31$.

Problem 13J / 3S. An object built from several identical cubes looks like '1' from behind, and like '3' from above (see the picture). How many cubes can be seen from the right if we know that the object contains maximum possible number of cubes?

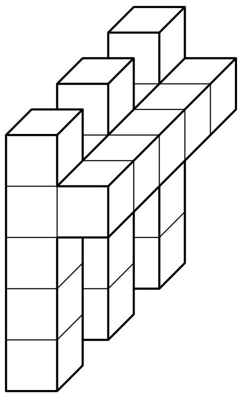


Note: The figure below shows a cube, its back view and its top view, respectively.

Result. 17

Solution. The structure can obviously fit in a box that is two cubes wide, five cubes high and five cubes long. Let us divide it in half and analyse its two $1 \times 5 \times 5$ parts separately. If we look at the structure from the front, we can see mirror '1'. Hence in the right part, there is one square visible from the front and five squares visible from above. The only way to achieve that is to put five cubes in one row. Similarly for the left part, there are five squares visible from the front and three squares visible from above, so we get the maximum number of cubes if we place five cubes in each of the three columns.

The resulting structure looks like the one in the following picture. If we look at it from the right, we can see 17 cubes.



Problem 14J / 4S. We say that a positive integer n is *delicious* if the sum of its digits is divisible by 17 and the same holds for $n+10$. What is the smallest delicious number?

Result. 7999

Solution. Denote by $Q(r)$ the sum of digits of r. If the tens digit of n differs from 9, then we have $Q(n+10) = Q(n)+1$. Hence the tens digit of n has to be 9. If the hundreds digit differs from 9, we have $Q(n+10) = Q(n) - 8$, but if the hundreds digit is 9 and the thousands digit is not, we get $Q(n+10) = Q(n) - 17$, so we can make both $Q(n)$ and $Q(n+10)$ divisible by 17. To keep n as small as possible, suppose that this is the case and $Q(n) = 2 \cdot 17 = 34$. The sum of all digits but the hundreds and tens one is therefore $34 - 2 \cdot 9 = 16 < 2 \cdot 9$, which means that two more digits are enough. Now it is easy to see that $n = 7999$ is the desired result.

Problem 15J / 5S. A bus company runs a line between towns A and D with stops in towns B and C (in this order). The ticket price is directly proportional to the distance travelled by the bus. For example, the ticket from A to C costs the same as the tickets from A to B and from B to C together. Moreover, the company does not offer return tickets, only one way ones. Lisa, a keen bus ticket collector, wants to gather tickets with all possible prices regardless of the direction of the journey. So far she has got the tickets costing 10, 40, 50, 60, and 70. What are the possible prices of the missing ticket?

Result. 20, 110

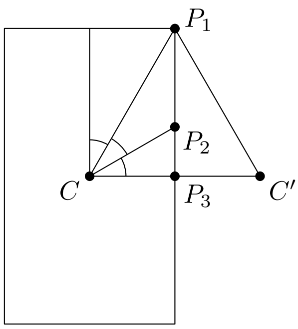
Solution. Assume first that Lisa has the most expensive ticket (i.e. the one from A to D) in her collection; thus its price is 70. As this price is the sum of the tickets for the sections AB, BC, and CD, at least two of which are already in Lisa's possession, we see that the only possibility for the prices of these three tickets is 10, 20, and 40, so the missing price has to be 20. It is easy to see that these prices can be assigned to the sections so that they agree with the rest of the tickets.

If the most expensive ticket is missing, then the ticket costing 70 has to be for a journey with one stop; the only way to decompose this price to a sum of two which Lisa already has is $10 + 60$. We infer that the ticket for the remaining section costs 40 and the longest journey costs $10 + 40 + 60 = 110$. It is again easy to check that these values are satisfactory.

Problem 16J / 6S. In a clock and watch store, Helen admires a watch which is packed in a transparent rectangular box such that the center of the box and the center of the watch (the point where the hands meet) coincide. The shorter side of the box is 3 cm long. She notices that at noon, the hour hand points to the middle of the shorter side of the box, and at one o'clock, it points to the corner of the box. How far apart are the two points on the boundary of the box where the hour hand points to one o'clock and to two o'clock, respectively?

Result. $\sqrt{3}$ cm

Solution. Denote by P_x the point on the boundary of the box where the hour hand points at x o'clock, and let C be the center of the box. Since $\angle P_2CP_1 = \angle P_3CP_2 = 30°$, we see that the point P_2 is the incenter and centroid of the equilateral triangle $CC'P_1$, where C' is C reflected through P_3. Now, the distance P_1P_2, which we are looking for, is 2/3 of the height of an equilateral triangle of side length 3 cm, i.e. $P_1P_2 = \frac{2}{3} \cdot \frac{1}{2} \cdot \sqrt{3} \cdot 3$ cm $= \sqrt{3}$ cm.



Problem 17J / 7S. Find an arrangement of $1, 2, \ldots, 9$ as a nine-digit number such that any two consecutive digits form a number which is a product $k \cdot l$ of digits $k, l \in \{1, 2, \ldots, 9\}$.

Result. 728163549

Solution. Let $x, y \in \{1, 2, \ldots, 9\}$ be distinct digits. A pair xy will be called *valid* whenever there exist $k, l \in \{1, 2, \ldots, 9\}$ such that $10x + y = kl$ holds. Since the only valid pair involving 9 is 49, the block 49 must appear at the end of the desired nine-digit number z. There are two valid pairs involving 7, namely 27 and 72. They cannot occur simultaneously, and the end of z is already occupied. Therefore 72 must be put at the beginning of z. Since valid pairs involving 8 are 18, 28, 48 and 81, and 4 has been already used to form 49, the only option is to form a block 281. Hence $z = 7281\ldots49$. Now we have to handle the remaining digits 3, 5 and 6. Since neither 13 nor 34 is a valid pair, and the only valid pair xy such that $x \in \{5, 6\}$ and $y = 3$ is 63, we find $z = 728163549$. Clearly, z satisfies all imposed conditions.

Problem 18J / 8S. Find the largest prime p less than 210 such that the number $210 - p$ is composite.

Note: Recall that number 1 is neither prime nor composite.

Result. 89

Solution. Instead of looking for the largest prime p, let us look for the smallest composite number n such that $210 - n$ is a prime. We have $210 = 2 \cdot 3 \cdot 5 \cdot 7$. Consequently, if n was divisible by 2, 3, 5, or 7, so would be $210 - n$, i.e. $210 - n$ would not be prime. The next smallest composite candidate for n is 11^2 which gives $p = 210 - 121 = 89$, which is a prime.

Problem 19J / 9S. The management of an elementary school decided to buy a certain number of pencils and to distribute them among the first grade pupils, who were divided into classes A, B, and C. If they gave the same number of pencils to every pupil, each of them would get nine pencils. If they handed them only to class A, each pupil in this class would get 30 pencils, and if they chose class B instead, each would get 36 pencils. How many pencils would a single pupil from class C receive if only this class was given pencils?

Result. 20

Solution. Denote by T the total number of pencils and by a, b, c the numbers of pupils in the corresponding classes. From the statement we know that $T = 9(a+b+c)$, $T = 30a$, and $T = 36b$. We are looking for T/c. Substituting $a = T/30$ and $b = T/36$ in the first equation yields

$$\begin{aligned} T &= \tfrac{3}{10}T + \tfrac{1}{4}T + 9c, \\ \tfrac{9}{20}T &= 9c, \\ \tfrac{T}{c} &= 20. \end{aligned}$$

Problem 20J / 10S. Find all four-digit square numbers which have the property that both the first two digits and the last two digits are non-zero squares, the latter one possibly starting with zero.

Result. 1681

Solution. Since for $k \geq 50$ the inequality $(k+1)^2 - k^2 > 100$ holds, $50^2 = 2500$ is the only square starting with 25 (similarly for 3600, 4900, 6400, and 8100). Hence the first two digits have to be 16. The only square greater than 1600 and less than 1700 is $41^2 = 1681$, which clearly satisfies the problem conditions.

Problem 21J / 11S. A driver was driving on a highway between two cities at a constant speed. Unfortunately, some parts of the highway were under repair, so he had to reduce his speed by one fourth while he was driving through these sections. Consequently, by the time he would normally have reached his destination, he had travelled only six sevenths of the entire distance. At that point, what fraction of the total time he had been driving did he spend in the sections under repair?

Result. 4/7

Solution. Let x be the fraction of time spent in the sections under repair. Then $1 - x$ is the fraction of time spent on the rest of the highway. Thus we have

$$\frac{6}{7} = \frac{3}{4}x + 1 - x,$$

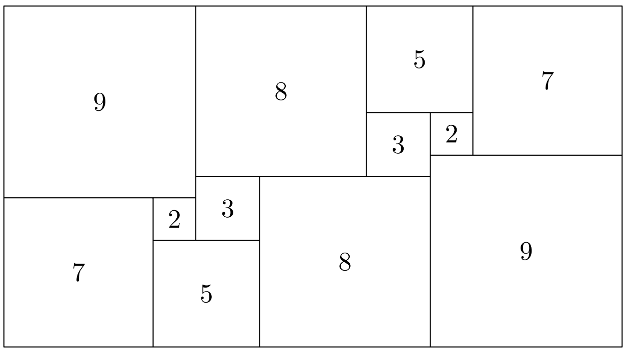
which yields $x = 4/7$.

Problem 22J / 12S. A rectangle with integral side lengths is decomposed into twelve squares with the following side lengths: 2, 2, 3, 3, 5, 5, 7, 7, 8, 8, 9, 9. What is the perimeter of the rectangle?

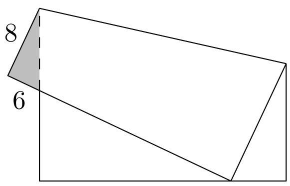
Result. 90

Solution. By summing the areas of the squares, the area of the rectangle is $464 = 2^4 \cdot 29$. The lengths of both sides of the rectangle have to be at least 9 because of the square with side length 9. Hence the only possible factorization is $16 \cdot 29$, which yields the perimeter 90.

Note: Such a decomposition exists, as one can see in the figure:



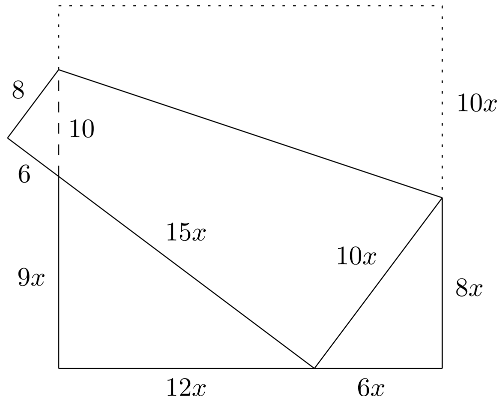
Problem 23J / 13S. A square sheet of paper is folded so that one of its vertices is precisely on one of the sides. As in the picture, there is a small triangle, which overlaps the original square. The length of its outer side that is adjacent to the line of the folding is 8 cm, and the length of the other outer side is 6 cm.



What is the side length of the paper?

Result. 36 cm

Solution. By checking angles, we see that all triangles in the statement picture are right-angled and similar to each other. If we denote the side lengths of the triangle in the lower right corner by $6x$, $8x$, and due to the Pythagorean theorem $10x$, then by "unfolding" we see that the side length of the square is $18x$. This implies that the lower left triangle has one side of length $18x - 6x = 12x$, as can be seen in the following picture. The other sides of this triangle are of length $9x$ and $15x$ due to the scaling factor $3/2$ of similarity. Now we can see that the side length of the given square is also $15x + 6$ cm, which yields $x = 2$ cm. Therefore the side length of the paper is 36 cm.



Problem 24J / 14S. An old steamship is moving along a canal at a constant speed. Simon wants to find out the length of the ship. While the ship is slowly advancing, he walks at the waterside at a constant speed from the rear of the ship to its nose counting 240 steps. Then he immediately turns around and walks backwards up to the rear of the steamship counting 60 steps. What is the length of the steamship in steps?

Result. 96

Solution. When Simon returns to the back of the ship, he has made 300 steps and the ship has moved on $240 - 60 = 180$ steps. Hence during the time in which Simon makes 60 steps, the ship moves $180 : 5 = 36$ steps forward. However, Simon reaches the back of the ship after 60 steps, so the length of the ship must be $60 + 36 = 96$ steps.

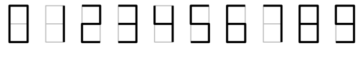
Problem 25J / 15S. The number $137641 = 371^2$ is the smallest six-digit number such that it is possible to cross out three pairwise different digits to obtain its square root: $\cancel{1}37\cancel{6}\cancel{4}1$. Find the largest six-digit number having this property.

Result. $992016 = 996^2$

Solution. Let $(1000 - n)^2$ be the desired number ($n \geq 1$). We can calculate the squares for $n = 1, 2, 3, 4, \ldots$ by using the formula $(1000 - n)^2 = 1000 \cdot (1000 - 2n) + n^2$ to get $999^2 = 998001$, $998^2 = 996004$, $997^2 = 994009$, $996^2 = 992016$, $\ldots$ It remains to see that the digits 2, 0, 1 are pairwise different and that $99\cancel{2}\cancel{0}\cancel{1}6 = 996$ is the square root of 992016. Therefore 992016 is the desired number.

Problem 26J / 16S. Linda typed something on her calculator, and so a three-digit number appeared on the display. Patrick, who was sitting opposite to her, noticed that from his point of view (upside-down) it looked exactly as a three-digit number that is greater by 369 than the one typed. What was the number that Linda typed?

Note: The calculator has a seven-segment display, therefore the digits look like this:



Result. 596

Solution. Notice that if we turn a digit upside-down, then either it does not change (0, 2, 5, 8), or it changes ($6 \leftrightarrow 9$), or we do not get a digit at all (1, 3, 4, 7). Let us denote by x the Linda's number and by x' the same number upside-down. Since the number 369 ends with 9, there are three candidates for the units digit of x: 0, 6, and 9 (all other admissible numbers lead to a non-admissible units digit of x').

If x ended with 0, then x' would begin with 0, which is impossible. Furthermore, x ending with 9 would result in x' ending with 8, i.e. x beginning with 8 and x' beginning with 6, which is a contradiction since $x > x'$. Finally, if 6 is the units digit of x, we have $x = \overline{5a6}$ and $x' = \overline{9a'5}$ for some admissible digit a and its upside-down version a'. However, $a = a'$ is not possible in the light of $x' - x = 369$, so a has to be either 6 or 9. Direct computation shows that $x = 596$ is the sought value.

Problem 27J / 17S. There are three families living on the island of Na-boi, each having two sons and two daughters. In how many ways can these twelve people form six married couples if the marriages of siblings are forbidden?

Result. 80

Solution. Denote the families A, B, and C. If the sons from A marry two sisters from some other family (w.l.o.g. B), then the daughters from A have to marry the sons from C (otherwise at least one son from C would marry his sister). It follows that the sons from B marry the daughters from C. This yields $2 \cdot 2^3 = 16$ ways—there are two ways to match the families, and then each pair of daughters has two options of marrying the sons from the matched family.

If, on the other hand, the sons from A decide to have wives from different families, their sisters must do the same in order to prevent inter-family marriages. The two sons together have eight choices of wives and the same holds for the daughters. After this matching is done, in each family B and C there is exactly one son and one daughter left unmarried, which gives the one and only way to complete the marriage scheme. Therefore this scenario gives $8 \cdot 8 = 64$ ways.

We conclude that there are $16 + 64 = 80$ ways to marry the people.

Problem 28J / 18S. Hansel and Gretel have baked a very big pizza consisting of 50 slices of equal circular sector shape. They have distributed olives on the slices in such a way that there are consecutively $1, 2, 3, \dots, 50$ olives placed clockwise on the slices. Now they want to divide the pizza into two equal halves by making a straight cut between the slices so that Hansel gets twice as many olives as Gretel. Determine the total number of olives on the four slices adjacent to the cutting line.

Result. 68, 136

Solution. Let us number the slices by the number of olives put on them. Note that the cutting line cannot separate the slices 1 and 50 (or the slices 25 and 26) because clearly

$$2 \cdot (1 + 2 + \dots + 25) < 26 + 27 + \dots + 50.$$

Hence we can assume that n, $n+1$, $n+25$, $n+26$ are the numbers of the slices adjacent to the cutting line, where $1 \le n \le 24$. The sum of these numbers is $4n+52$. Now we compute $(n+1)+(n+2)+\dots+(n+25) = 25n + \frac{1}{2} \cdot 25 \cdot 26 = 25(n+13)$ and $1 + 2 + \dots + 50 = \frac{1}{2} \cdot 50 \cdot 51 = 25 \cdot 51$. Then we have two possibilities:

$$25(n+13) = \frac{1}{3} \cdot 25 \cdot 51 = 25 \cdot 17 \quad \text{or} \quad 25(n+13) = \frac{2}{3} \cdot 25 \cdot 51 = 25 \cdot 34.$$

The first possibility yields $n = 4$ and the desired sum is $4n + 52 = 68$. From the second possibility we obtain $n = 21$ and $4n + 52 = 136$. In total, there are two solutions, namely 68 and 136.

Problem 29J / 19S. Find all primes p with the property that $19p + 1$ is the cube of an integer.

Result. 421

Solution. If p is a prime fulfilling the given condition, then there exists an integer $k > 2$ with $k^3 = 19p + 1$. Therefore we get the equation

$$19p = k^3 - 1 = (k-1)(k^2 + k + 1).$$

Due to $k > 2$, both factors on the right-hand side are proper divisors of $19p$. Since $19p$ is a product of two primes, we have either $k - 1 = 19$ or $k^2 + k + 1 = 19$. The first case leads to $k = 20$ and $p = 400 + 20 + 1 = 421$, which is a prime. In the second case, the quadratic equation $k^2 + k - 18 = 0$ has no integer solution. Therefore $p = 421$ is the only solution.

Problem 30J / 20S. In a parallelogram $ABCD$, the point E lies on side AD with $2 \cdot AE = ED$ and the point F lies on side AB with $2 \cdot AF = FB$. Lines CF and CE intersect the diagonal BD in G and H, respectively. Which fraction of the area of parallelogram $ABCD$ is covered by the area of the pentagon $AFGHE$?

Result. $\frac{7}{30}$

Solution. In the following, we shall denote the area by square brackets. Since the triangles EHD and CHB are similar, the equation

$$\frac{BH}{HD} = \frac{BC}{ED} = \frac{AD}{\frac{2}{3} \cdot AD} = \frac{3}{2}$$

follows. Due to the similarity of the triangles FBG and CDG, we get $\frac{DG}{GB} = \frac{3}{2}$. Therefore $DH = BG = \frac{2}{5} \cdot DB$ and $HG = \frac{1}{5} \cdot DB$. Since

$$[ECD] = \frac{2}{3}[ACD] = \frac{2}{3} \cdot \frac{1}{2}[ABCD] = [FBC],$$

we get

$$[AFGHE] = [AFCE] - [GCH] = \left(\frac{1}{3} - \frac{1}{5} \cdot \frac{1}{2}\right)[ABCD] = \frac{7}{30}[ABCD].$$

Problem 31J / 21S. Find the largest five-digit number with non-zero digits and the following properties:

- The first three digits form a number which is 9 times the number formed by the last two digits.
- The last three digits form a number which is 7 times the number formed by the first two digits.

Result. 85595

Solution. Let $\overline{abcde}$ be a five-digit number such that $\overline{abc} = 9 \cdot \overline{de}$ and $\overline{cde} = 7 \cdot \overline{ab}$. Then we have

$$63 \cdot \overline{de} = 7 \cdot \overline{abc} = 70 \cdot \overline{ab} + 7c = 10 \cdot \overline{cde} + 7c = 1007c + 10 \cdot \overline{de},$$

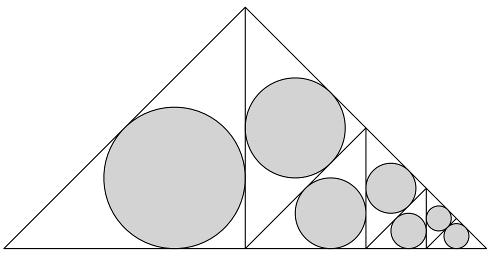
so $\overline{de} = \frac{1007c}{53} = 19c$. Analogously we get $\overline{ab} = 17c$. If $c \geq 6$, then the numbers $17c$ and $19c$ are greater than 100. It follows that the maximum possible value of c is 5, i.e. the maximum possible value of $17119c$ is 85595.

Problem 32J / 22S. Twelve smart men, each of whom was randomly dealt one of twelve cards—nine blank cards and three special cards labelled J, Q, and K—were sitting in a circle. Everybody looked at his card, and then passed it to his right-hand neighbour. Continuing in this way, after every look at a card, they were asked to raise their hands simultaneously if they knew who was holding which special card at that very moment. Nobody raised his hand after having seen four cards. Exactly one man raised his hand after having seen his fifth card. Next, x men raised their hand after having seen six cards and y men after seven cards. Determine xy.

Result. 42

Solution. The first man to raise his hand was the first one to see all three special cards. This man must have received one special card as his fifth card because otherwise he could have raised his hand earlier. Moreover, he must have received one special card at the very beginning because otherwise his left-hand neighbour would have seen all special cards after the previous round. Denote C_1 the first card, C_3 the fifth card, and C_2 the remaining special card that the man got in the meantime. After the subsequent looks at their cards, exactly the persons who encountered C_2 and at least one other card know or can deduce the positions of all three special cards—the positions of C_1 and C_3 are known to everyone, but their labels are not. It is easy to see that there are six such men after having seen six cards and seven after having seen seven cards, so the answer is 42.

Problem 33J / 23S. Within an isosceles right-angled triangle with the length of the base 1, seven circles are constructed as in the picture:



What is the total area of the circles?

Result. $\pi\frac{3-2\sqrt{2}}{4} = \pi\frac{(1-\sqrt{2})^2}{4} = \pi\frac{1}{4(1+\sqrt{2})^2} = \pi\frac{1}{4(3+2\sqrt{2})}$

Solution. If an isosceles right-angled triangle is cut into two equal triangles, then they are similar to the original one with the ratio $1 : \sqrt{2}$, and so are the radii of the incircles. Thus each new incircle has half the area of the original one. In other words, the total area of the incircles is not changed when cutting the big triangle. Hence it is sufficient to determine the area of the incircle of the big triangle, radius of which we compute using equal tangents. Since the length of the legs of the big triangle equals $\frac{\sqrt{2}}{2}$ by Pythagorean theorem and the perpendiculars from the incenter to the legs form a square, we get that the radius is $\frac{\sqrt{2}}{2} - \frac{1}{2}$, so the area equals

$$\pi\frac{3-2\sqrt{2}}{4}.$$

Problem 34J / 24S. Find all primes p such that $p + 11$ divides $p(p+1)(p+2)$.

Result. 7, 11, 19, 79

Solution. As p is a prime, it is either equal to 11 (which clearly satisfies the given condition) or coprime to $p + 11$. In the latter case, the product in question is divisible by $p + 11$ if and only if $(p+1)(p+2)$ is. This reduced product modulo $p + 11$ equals $(-10) \cdot (-9)$, thus $p + 11 \mid 90$. This is satisfied for $p \in \{7, 19, 79\}$.

Problem 35J / 25S. Wood lice (Porcellio scaber) have fourteen legs. Mother wood louse has large supplies of identical socks and shoes and prepares her children for the upcoming cold season. She explains to little wood louse Jim that he may put on his socks and shoes in arbitrary order, but he has to bear in mind that on each individual leg he has to put a sock prior to a shoe. In how many ways can little Jim proceed to dress his feet?

Result. $\frac{28!}{2^{14}}$

Solution. A way to dress Jim's legs can be represented by a 28-tuple consisting of 14 socks and 14 shoes in which the sock belonging to a specific leg is in a position previous to that of the shoe belonging to the same leg. For the first pair of a sock and a shoe, there are $\binom{28}{2}$ possibilities. For the second pair, there are $28-2=26$ places, hence $\binom{26}{2}$ possibilities. Continuing this way, there remains only $\binom{2}{2}=1$ possibility for the last pair. Therefore the result is

$$\binom{28}{2}\cdot\binom{26}{2}\cdot\binom{24}{2}\cdots\binom{2}{2}=\frac{28!}{2^{14}}.$$

Problem 36J / 26S. Let x be a real number such that $x^3+4x=8$. Determine the value of x^7+64x^2.

Result. 128

Solution. It suffices to plug in $x^3=8-4x$ into the given expression as follows:

$$x^7+64x^2=x\cdot(x^3)^2+64x^2=x(8-4x)^2+64x^2=64x+16x^3=16(x^3+4x)=128.$$

Problem 37J / 27S. In an isosceles triangle ABC with base AB, let D be the intersection of angle bisector of $\angle ACB$ with AB and E the intersection of angle bisector of $\angle BAC$ with BC. Given that $AE=2\cdot CD$, find $\angle BAC$.

Result. 36°

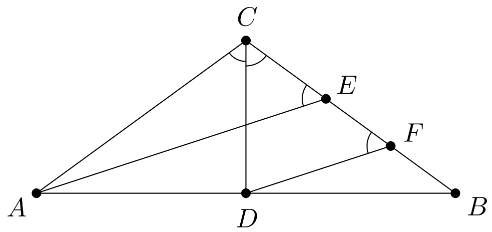
Solution. Let F be the point on BC such that $AE \parallel DF$. Now $DF=\frac{1}{2}AE=CD$, so the triangle FCD is isosceles with base FC. Denote $\varphi=\angle BAC$. Then

$$\angle AEC=\angle DFC=\angle FCD=\angle DCA=90^\circ-\varphi.$$

Finally, since $\angle CAE=\frac{1}{2}\varphi$, we have (using $\triangle AEC$)

$$\frac{1}{2}\varphi+3(90^\circ-\varphi)=180^\circ,$$

so $\varphi=36^\circ$.



Problem 38J / 28S. Given a sequence of real numbers (a_n) such that $a_1=2015$ and $a_1+a_2+\cdots+a_n=n^2\cdot a_n$ for each $n\ge 1$, find a_{2015}.

Result. $\frac{1}{1008}$

Solution. By subtracting the recursive formulas for n and $n-1$, we get $a_n=n^2\cdot a_n-(n-1)^2\cdot a_{n-1}$, which can be simplified to $a_n=\frac{n-1}{n+1}a_{n-1}$. Thus

$$a_n=\frac{n-1}{n+1}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdots\frac{2}{4}\cdot\frac{1}{3}\cdot a_1=\frac{2a_1}{n(n+1)},$$

and therefore $a_{2015}=\frac{2\cdot 2015}{2015\cdot 2016}=\frac{1}{1008}$.

Problem 39J / 29S. Kate and Harriet invented the following game: They colour the faces of two twelve-sided fair dice with cyan, magenta, and yellow so that each colour is present on at least one face of each die, and there are exactly four yellow faces on the first die. If they throw the dice and both of these show the same colour, Kate wins, otherwise Harriet wins. Suppose that the colours are distributed so that the girls have equal chances of winning. How many magenta faces must be there on the second die?

Result. 1, 9

Solution. Denote by c_1, m_1, y_1, c_2, m_2, y_2 the numbers of faces of respective colours on the dice. We know that $c_1 + m_1 + y_1 = 12$, $c_2 + m_2 + y_2 = 12$, and $y_1 = 4$. Moreover, from the total of $12^2 = 144$ possible outcomes of throwing the dice, exactly one half results in same colours, so

$$c_1c_2 + m_1m_2 + y_1y_2 = 72.$$

Expressing the left-hand side in terms of c_1, c_2, and m_2 using the relations above, we obtain

$$c_1c_2 - c_1m_2 - 4c_2 + 4m_2 + 48 = 72,$$

or

$$(c_1 - 4)(c_2 - m_2) = 24.$$

Using $-3 \leq c_1 - 4 \leq 3$ and $-9 \leq c_2 - m_2 \leq 9$, we deduce that $c_2 - m_2$ is either 8 or -8, which together with $0 < y_2 = 12 - c_2 - m_2$ yields that m_2 is either 1 or 9. A straightforward check shows that both these values are possible.

Problem 40J / 30S. There are $n > 24$ women sitting around a great round table, each of whom either always lies or always tells the truth. Each woman claims the following:

- She is truthful.
- The person sitting twenty four seats to her right is a liar.

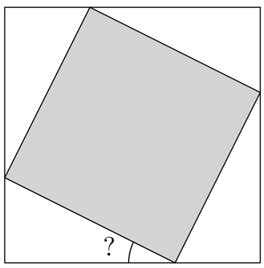
Find the smallest n for which this is possible.

Result. 32

Solution. Let us pick a woman, then the one 24 seats to her right, then the next one 24 seats to the right etc. After a certain number of such steps, say s, we get back to the original woman. This s is readily seen to be the smallest positive integer such that $24s$ is a multiple of n. Hence $s = n/d$, where d is the greatest common divisor of n and 24.

Observe that the truthfulness of the women has to alternate, i.e. every woman is truthful if and only if the one 24 seats to her right is a liar. If s were odd, this would lead to a contradiction. Therefore n has to be divisible by a higher power of 2 than 24 is. The smallest candidate for n is 32, and it is easy to check that this number satisfies the given conditions.

Problem 41J / 31S. Two squares have a common center and the vertices of the smaller one lie on the sides of the bigger one. If the small square is removed from the big one, it decomposes into four congruent triangles, the area of each being one twelfth of the area of the big square. What is the size in degrees of the smallest internal angle of the triangles?



Result. 15°

Solution. Denote by a, b the legs and by c the hypotenuse of the remaining triangles, and assume $a \leq b$. A straightforward computation shows that the area of the small square equals two thirds of the area of the big square. Hence the area of one of the triangles is one eighth of the area of the small square, that is

$$\frac{1}{2}ab = \frac{1}{8}c^2.$$

Let α be the smallest internal angle in the triangle. Then $a = c \sin \alpha$ and $b = c \cos \alpha$, so

$$c^2 = 4ab = 4c^2 \sin \alpha \cos \alpha = 2c^2 \sin 2\alpha,$$

or

$$\sin 2\alpha = \frac{1}{2}.$$

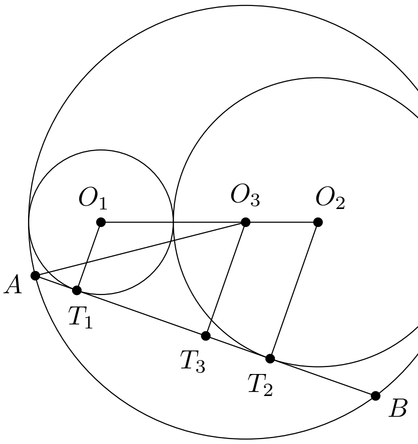
Since $\alpha \leq 45°$, it follows that $2\alpha = 30°$, and finally, $\alpha = 15°$.

Problem 42J / 32S. A circle ω_3 of radius 3 is internally tangent to circles ω_1, ω_2 of radii 1 and 2, respectively. Moreover, ω_1 and ω_2 are externally tangent. Points A, B on ω_3 are chosen so that the segment AB is a common external tangent of ω_1 and ω_2. Find the length of AB.

Result. $\frac{4}{3}\sqrt{14}$

Solution. Denote O_1, O_2, O_3 the centers of ω_1, ω_2, ω_3, respectively, and let T_1, T_2, T_3 be the feet of perpendiculars from O_1, O_2, O_3 to the segment AB (hence T_1 and T_2 are the points of tangency of AB to ω_1, ω_2). Since $O_1T_1 \parallel O_2T_2 \parallel O_3T_3$, $O_1T_1 = 1$, $O_2T_2 = 2$, and $O_1O_3 = 2O_2O_3$, where O_1, O_2, O_3 are collinear, we easily get $O_3T_3 = \frac{5}{3}$ (consider similar triangles). Applying the Pythagorean theorem to the triangle AO_3T_3, we conclude that

$$AB = 2AT_3 = 2\sqrt{3^2 - \left(\frac{5}{3}\right)^2} = \frac{4}{3}\sqrt{14}.$$



Problem 43J / 33S. One day, Oedipus, an intrepid hero, met the Sphinx, who gave him the following riddle: She chose a positive two-digit integer S. Oedipus could choose three one-digit integers $a < b < c$ and ask if S was divisible by them. For each of these numbers, he got an answer (either 'yes' or 'no'). Oedipus fell into despair as there were exactly two numbers satisfying the divisibility conditions. But then the Sphinx told him that she had been wrong about the divisibility by b, which fortunately allowed Oedipus to find S with certainty. What was the value of S?

Result. 84

Solution. There are clearly three two-digit numbers satisfying both given answers about divisibility by a and c (two corresponding to the Sphinx' first answer and one after the correction). Moreover, both these answers have to be positive since a negative answer would result in more than three eligible two-digit numbers. Hence these numbers are multiples of $\operatorname{lcm}(a, c) = m$. It follows that $25 \leq m \leq 33$, but only two numbers within this range are the least common multiples of two digits, namely $28 = \operatorname{lcm}(4, 7)$ and $30 = \operatorname{lcm}(5, 6)$. The latter is impossible because of $c - a \geq 2$, so we have $a = 4$ and $c = 7$. Now if b were 5, then there would be no two-digit number being divisible by a, b, and c simultaneously. In the case $b = 6$, the negative answer about b corresponds to the numbers 28 and 56, while the positive answer yields $S = 84$.

Problem 44J / 34S. Four people were moving along a road, each of them at some constant speed. The first one was driving a car, the second one was riding a motorcycle, the third one was riding a Vespa scooter, and the fourth one a bicycle. The car driver met the Vespa at 12 noon, the bicyclist at 2 p.m., and the motorcyclist at 4 p.m. The motorcyclist met the Vespa at 5 p.m., and the bicyclist at 6 p.m. At what time did the bicyclist meet the Vespa scooter?

Result. 3:20 p.m.

Solution. Since the time in question does not depend on the chosen reference frame, we may assume that the car is not moving at all. Under this assumption, the motorcycle needed one hour from where it met the car to its meeting point with the Vespa, whereas the Vespa needed five hours for the same distance, thus the motorcycle was five times faster. Similarly, one may deduce that the motorcyclist was twice as fast as the bicyclist, hence the ratio of the speeds of the Vespa and the bicycle was $2 : 5$.

If the Vespa needed t hours to get from the car to its meeting point with the bicyclist, then the bicyclist needed $t - 2$ hours. The ratio of these required times equals the inverse of the ratio of the speeds, so

$$\frac{t-2}{t} = \frac{2}{5},$$

or $t = 10/3$. Finally, using the fact that the Vespa met the car at 12 noon, we infer that it met the bicyclist at 3:20 p.m.

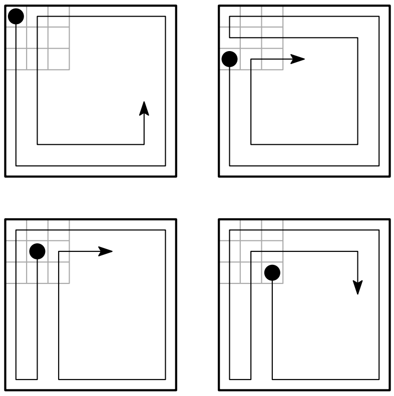
Problem 45J / 35S. The floor of a hall is covered with a square carpet of side length twenty-two meters. A robotic vacuum cleaner (robovac) is given the task to clean the carpet. For its convenience, the carpet is divided into 484 unit squares. The robovac cleans one square after another in accordance with the following rules:

- Once it has vacuumed a square, it must not move onto it again.
- It keeps moving in one direction unless it is forced to change it by approaching the edge of the carpet or a previously cleaned square.
- If it has to change the direction and there are two options, it may choose whichever it prefers.

At the beginning, the robovac is placed on one square and it may choose any admissible direction. For how many starting squares is the robovac able to clean the whole carpet if it does not need to finish its work on the edge?

Result. 20

Solution. If the robovac does not start in one of the corner 3×3 'multisquares', there always remains a non-cleaned part of the carpet—when the robovac leaves the edge of the carpet (i.e. no later than in its seventh move), it divides the remaining non-cleaned squares into two separate regions. We can routinely verify that the squares with coordinates $(1, 2)$, $(2, 1)$, $(2, 3)$, $(3, 2)$ and their symmetric counterparts in the other corners do not satisfy the given conditions for similar reasons. However, those with coordinates $(1, 1)$, $(2, 2)$, $(3, 3)$, $(3, 1)$, $(1, 3)$ do. Thus there are $4 \cdot 5 = 20$ possible starting squares in total.



Problem 46J / 36S. Let the points A, B, C, D, E, F lie clockwise in this order on a circle ω. Assume further that AD is a diameter of ω, BF intersects AD and CE in G and H, respectively, $\angle FEH = 56°$, $\angle DGB = 124°$, and $\angle DEC = 34°$. Find $\angle CEB$.

Result. $22°$

Solution. The inscribed angle theorem implies that $\angle CEB = \angle CDB$, so let us compute $\angle CDB$. Since the quadrilateral $BCEF$ is cyclic, we have

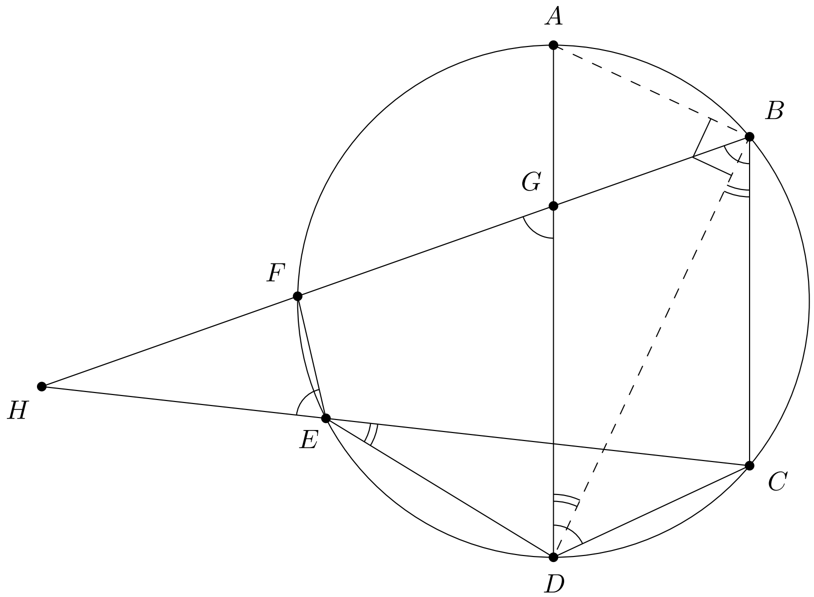
$$\angle FBC = 180° - \angle FEC = 180° - (180° - \angle FEH) = 56°.$$

Moreover, $\angle DGF = 180° - \angle DGB = 56°$, so AD and BC are parallel. This implies $\angle ADB = \angle DBC$ and using inscribed angles we further infer $\angle DBC = \angle DEC = 34°$. Since $\angle ABD = 90°$ (Thales' theorem) and trapezoid $ABCD$ is cyclic, we obtain

$$\angle ADC = 180° - \angle ABC = 180° - (\angle ABD + \angle DBC) = 56°,$$

and finally

$$\angle CEB = \angle CDB = \angle ADC - \angle ADB = 56° - 34° = 22°.$$



Problem 47J / 37S. Ten people—five women and their husbands—took part in E events. We know that no married couple took part in the same event, every pair of non-married people (including same-sex pairs) took part in exactly one event together, and one person attended only two events. What is the smallest E for which this is possible?

Result. 14

Solution. Denote the couples (a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2), (e_1, e_2). We may w.l.o.g. assume that a_1 is the person who attended only two events and that the first of these events was also attended by b_1, c_1, d_1, and e_1, whereas the second one by their counterparts. Every other event could have been attended by at most two people from $b_1, b_2, \ldots, e_1, e_2$ simultaneously, so there had to be at least 12 other events. If further a_2 takes part in any disjoint four of these events, we obtain the situation described in the statement, thus the smallest value of E is 14.

Problem 48J / 38S. Students were given a three-digit number $\overline{abc}$ with $0 < a < b < c$. The task was to multiply it by 6 and then swap the tens digit with the hundreds digit. Alex made a mistake and performed the swap before the multiplication, but the result turned out to be correct! Find $\overline{abc}$.

Result. 678

Solution. Since $0 < a < b$, we have $b \geq 2$ and $6 \cdot \overline{bac} > 1200$, so the result obtained by Alex must be a four-digit number, say $6 \cdot \overline{bac} = \overline{defg}$. On the other hand, we know that $6 \cdot \overline{abc} = \overline{dfeg}$. Subtracting these two equalities, we get

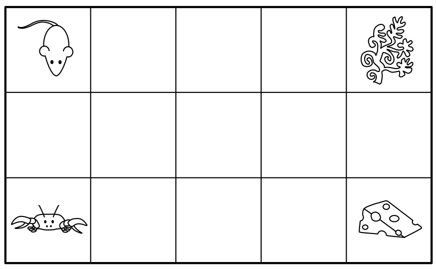
$$6(\overline{bac} - \overline{abc}) = \overline{defg} - \overline{dfeg}.$$

Thus $540(b - a) = 90(e - f)$, or $6(b - a) = e - f$. As e, f are digits, we have $e - f \leq 9$. It follows that $e - f = 6$ and $b - a = 1$ ($b - a > 0$ because $b > a$). Substituting $b = a + 1$ in $6 \cdot \overline{bac} = \overline{defg}$ gives

$$\overline{defg} = 6(100(a + 1) + 10a + c) = 660(a + 1) - 6(10 - c).$$

This means that $\overline{defg}$ must differ from some multiple of 660 by a non-zero multiple of 6 not exceeding 42 ($c \geq 3$). Remembering that $e - f = 6$, and considering successively $a = 1, 2, \ldots, 7$, we determine all candidates for the number $\overline{defg}$: 1938, 2604, 3930, 3936, 4602, 4608. Only for $\overline{defg} = 4608$ the number $\overline{bac} = \overline{defg}/6 = 768$ turns out to have distinct non-zero digits. Now we can directly check that indeed $6 \cdot \overline{abc} = 6 \cdot 678 = 4068 = \overline{dfeg}$.

Problem 49J / 39S. Consider a 5×3 grid. A mouse sitting in the upper left corner wants to reach a piece of cheese in the lower right corner, whereas a crab sitting in the lower left corner wants to reach algae in the upper right corner. They move simultaneously. Every second, the mouse moves one square right or down, and the crab moves one square right or up. In how many ways can the animals reach their food so that they do not meet at all?



Result. 70

Solution. Observe that the animals may meet only in the middle row. Furthermore, the path of each animal is completely determined by the visited squares in the middle row. It is easy to see that the animals do not meet if these middle parts of their paths do not intersect, so we are looking for the number of pairs of disjoint segments of the middle row.

Let us first handle the case when there is at least one unused square between the segments. Then the number of pairs may be computed as $2 \cdot \binom{6}{4}$, since one chooses four out of six edges of squares in the middle row and then chooses the animal which uses the segment between the first two of the chosen edges (the second animal uses the segment between the second two edges). If there is no gap between the segments, we obtain $2 \cdot \binom{6}{3}$ pairs since the segments are described by three edges.

In total, there are $2 \cdot \binom{6}{4} + 2 \cdot \binom{6}{3} = 70$ possible ways.

Problem 50J / 40S. Find the number of positive integers n not exceeding 1000 such that the number $\lfloor \sqrt[3]{n} \rfloor$ is a divisor of n.

Note: The symbol $\lfloor x \rfloor$ denotes the integral part of x, i.e. the greatest integer not exceeding x.

Result. 172

Solution. Observe that $\lfloor \sqrt[3]{n} \rfloor = k$ if and only if $k^3 \leq n \leq (k+1)^3 - 1$. Out of the $3k^2 + 3k + 1$ numbers in this range, every k-th is divisible by k starting with k^3, thus there are $3k + 4$ such numbers. It remains to sum this expression for all numbers k such that $(k+1)^3 - 1 \leq 1000$ (i.e. $k \leq 9$) and add one for the number 1000, which satisfies the given condition as well. Hence the desired result is

$$1 + \sum_{k=1}^{9} (3k + 4) = 1 + 9 \cdot 4 + 3 \cdot \frac{9 \cdot 10}{2} = 172.$$

Problem 51J / 41S. Find all real m such that the roots of the equation

$$x^3 - 15\sqrt{2}x^2 + mx - 195\sqrt{2} = 0$$

are side lengths of a right-angled triangle.

Result. 281/2

Solution. Let a, b, c be the roots of the given equation, which are also side lengths of a right-angled triangle. Assume w.l.o.g. that $0 < a, b < c$, hence by Pythagoras' theorem, the equation $a^2 + b^2 = c^2$ holds. By Vieta's formulas (or working out $(x-a)(x-b)(x-c)$ and then comparing coefficients), we get

$$15\sqrt{2} = a + b + c, \quad m = ab + ac + bc, \quad 195\sqrt{2} = abc.$$

Squaring $15\sqrt{2} - c = a + b$ leads to $450 - 30\sqrt{2}c = 2ab$. After multiplying by c and substituting $abc = 195\sqrt{2}$, we get the quadratic equation

$$\sqrt{2}c^2 - 15c + 13\sqrt{2} = 0,$$

having roots $c_1 = \sqrt{2}$ and $c_2 = 13\sqrt{2}/2$. Since the constraints $0 < a, b < c$ and $abc = 195\sqrt{2}$ allow only $c = 13\sqrt{2}/2$, the desired number m can be calculated via

$$m = ab + ac + bc = \frac{1}{2} \cdot ((a+b+c)^2 - 2c^2) = \frac{1}{2} \cdot 450 - c^2 = 281/2.$$

Problem 52J / 42S. There are green and red apples in a basket, at least one red and two green ones. The probability that a randomly chosen apple is red is 42 times higher than the probability that two randomly chosen apples (without replacement) are both green. How many green apples and how many red apples are there in the basket?

Result. 4 green and 21 red

Solution. Let g be the number of green apples and r the number of red apples. The statement may be rewritten as

$$\frac{r}{g+r} = 42 \cdot \frac{g \cdot (g-1)}{(g+r) \cdot (g+r-1)},$$

equivalently

$$r^2 + (g-1)r - 42g \cdot (g-1) = 0.$$

The latter can be viewed as a quadratic equation with variable r and parameter g . The discriminant

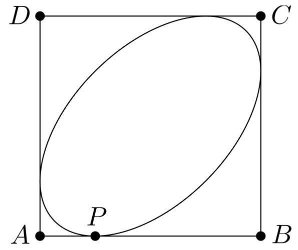
$$(g-1)^2 + 168g \cdot (g-1) = 169g^2 - 170g + 1$$

must be a square of an integer, otherwise the roots would be irrational. Due to $g \geq 2$, we obtain the inequalities

$$(13g-6)^2 = 169g^2 - 156g + 36 > 169g^2 - 170g + 1 > 169g^2 - 208g + 64 = (13g-8)^2.$$

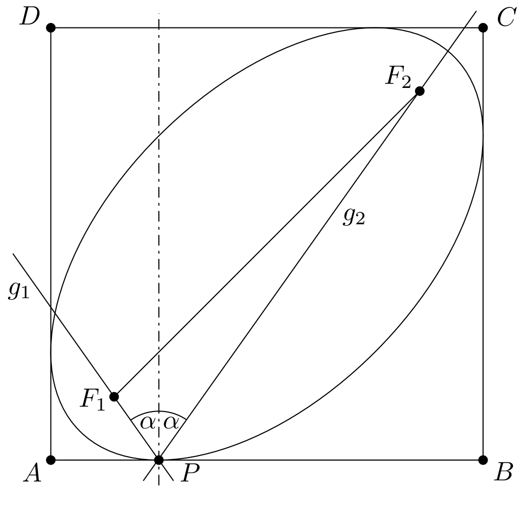
Therefore $169g^2 - 170g + 1$ has to be equal to $(13g-7)^2$, which implies $12g = 48$, or $g = 4$. Then the roots of the quadratic equation are -24 and 21, but since $r > 0$, we get $r = 21$ as the only solution. Hence there are 21 red and 4 green apples in the basket.

Problem 53J / 43S. Gilbert Bates, a very rich man, wants to have a new swimming pool in his garden. Since he likes symmetry, he tells the gardener to build an elliptical-shaped pool within a 10 m $\times$ 10 m square $ABCD$. This elliptical pool should touch all four sides of the square, particularly the side AB at point P which is 2.5 m away from A. The gardener, who knows very well how to construct an ellipse if the foci and a point on the ellipse are given, reflects that due to symmetry he only needs the distance of the foci in this case. Can you help him by computing the distance of the foci in meters?



Result. 10

Solution. We solve the problem in a more general way. Let $ABCD$ be a square of side length 1 with corner A in the origin of a coordinate system. Point P is on AB and has coordinates $(b, 0)$ with $0 < b < \frac{1}{2}$. If focus F_1 has coordinates (f, f), then focus F_2 has coordinates $(1-f, 1-f)$ (both foci have to lie on the diagonal AC symmetrically with respect to the other diagonal).



Now line g_1 through P and F_1 is given by

$$y = (x-b) \cdot \frac{f}{f-b},$$

and line g_2 through P and F_2 by

$$y = (x-b) \cdot \frac{f-1}{b+f-1}.$$

Since the line through P perpendicular to the tangent AB bisects $\angle F_2PF_1$, the gradient of g_2 has to be the negative of the gradient of g_1. Therefore we get

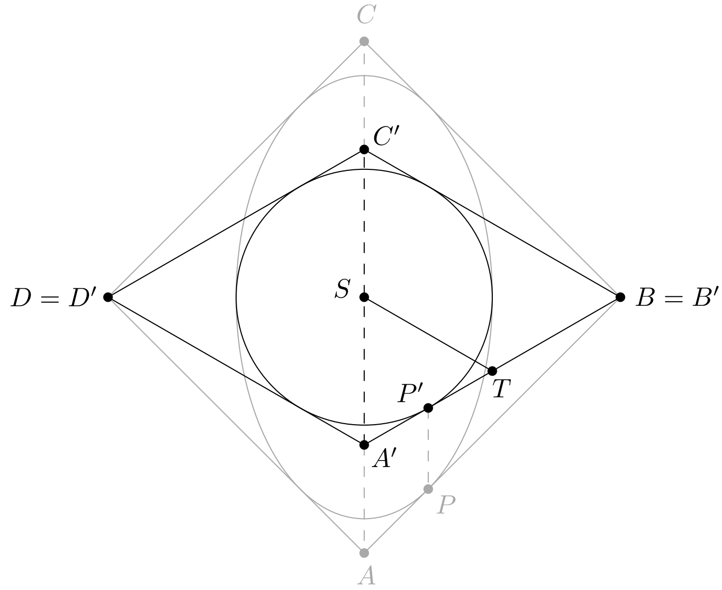
$$\frac{f}{f-b} = (-1) \cdot \frac{f-1}{b+f-1},$$

which can be simplified to

$$f^2 - f + \frac{b}{2} = 0.$$

This quadratic equation has two solutions $f_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1-2b}\right)$, each one corresponding to one focus of the ellipse. The distance of the foci can now be computed by Pythagoras' theorem, yielding $\sqrt{2-4b}$. Setting $b = \frac{1}{4}$ and scaling up by 10 gives the desired result $F_1F_2 = 10$.

Alternative solution. Let us shrink both the square and the ellipse along AC so that the ellipse becomes a circle and denote the new points with a prime. Further, let S be the midpoint of AC and T the midpoint of $A'B'$.



Since $A'P' = \frac{1}{4}A'B'$, we have $A'P' = P'T$, so $SA' = ST$. It follows that $A'C' = B'C'$ and that the triangle $A'B'C'$ is equilateral. Hence the factor of shrinking is

$$\frac{A'C'}{AC} = \frac{\sqrt{3}}{3}.$$

It is easy to compute that the radius of the circle (which coincides with the length b of the minor semi-axis of the ellipse) equals $\frac{1}{4}AC$. The length a of the major semi-axis is obtained by dividing the minor semi-axis by the factor of shrinking. Knowing the lengths a and b, it remains to compute the eccentricity using the formula $e = \sqrt{a^2 - b^2}$ and multiply it by two to get the distance of the foci.

Problem 54J / 44S. A sequence (a_n) is given by $a_1 = 1$, and $a_n = \lfloor\sqrt{a_1 + a_2 + \cdots + a_{n-1}}\rfloor$ for $n > 1$. Determine a_{1000}.

Note: The symbol $\lfloor x\rfloor$ denotes the integral part of x, i.e. the greatest integer not exceeding x.

Result. 495

Solution. By writing down the first few members $(1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 8, 9, 9, \dots)$, we discover that number 1 repeats four times, while numbers greater than 1 seem to appear only twice or thrice. Let us prove by induction that those appearing three times are exactly natural powers of 2.

Suppose we have written down the beginning of the sequence up to the first occurrence of n $(n > 1)$ and assume it behaves as described above. Let k be the largest integer such that $2^k < n$. Then the sum of all written members equals

$$s_1 = (1 + 2 + \cdots + n) + (1 + 2 + \cdots + n - 1) + (1 + 2 + 2^2 + \cdots + 2^k) + 1 = n^2 + 2^{k+1}.$$

Since $2^{k+1} = 2 \cdot 2^k < 2n < 2n+1 = (n+1)^2 - n^2$, we have $s_1 < (n+1)^2$, and hence the following member is $\lfloor\sqrt{s_1}\rfloor = n$.

Let us determine the next number. Now the sum is $s_2 = s_1 + n = n^2 + n + 2^{k+1}$. If $2^{k+1} < n + 1$, then again $s_2 < (n + 1)^2$ and the following member is n. However, k is the largest integer satisfying $2^k < n$, so it holds $2^{k+1} \geq n$. Therefore this case occurs exactly when $2^{k+1} = n$. If n is not a natural power of 2, then the next member is $n + 1$ because $n + 1 \leq 2^{k+1} < 2n < 3n + 4$, which is equivalent to $(n + 1)^2 \leq n^2 + n + 2^{k+1} < (n + 2)^2$.

It remains to show that if $n = 2^{k+1}$, then there comes $n + 1$ after three occurrences of n. This is easy because if we compute the next sum $s_3 = s_2 + n = n^2 + 2n + 2^{k+1} = n^2 + 3n$, we immediately get $(n + 1)^2 < s_3 < (n + 2)^2$. Thus the induction step is complete and we get $a_{1000} = 495$ (since $500 = a_{1010} = a_{1009}$).

Problem 55J / 45S. Determine the number of 4×4 tables with non-negative integer entries such that

- every row and every column contains at most two non-zero entries,
- for each row and column, the sum of its entries is 3.

Result. 576

Solution. Observe that every such table uniquely decomposes into an entrywise sum of two tables, the first one having exactly one 1 in every row and every column, and the second one exactly one 2 in every row and every column. Conversely, having such a pair of tables, adding them entrywise results in a table satisfying the given criteria. Therefore $(4!)^2 = 576$ is the desired result.