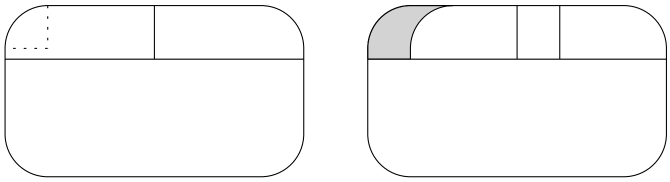
Problem 1. The windows on an old tram look like shown in the picture. All curves forming the round corners are arcs of a quarter circle with radius 10 cm. A portion of the sliding window is opened 10 cm as you can see in the second picture. The height of the open section is 13 cm. What is the area of the opening in cm^2?



Result. 130

Solution. It is just the area of the sliding window overlapping the fixed part, which is a rectangle 10 cm $\times$ 13 cm.

Problem 2. A rectangle is subdivided into nine smaller rectangles as shown in the picture. The number written inside a small rectangle denotes its perimeter. Find the perimeter of the large rectangle.

	9	
14	10	17
	12	

Result. 42

Solution. Looking at the picture we notice that the perimeter of the large rectangle equals the sum of the perimeters of the four outer small rectangles with given perimeter minus the perimeter of the middle one. Therefore, the answer is

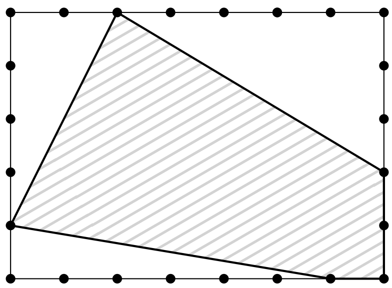
$$14 + 9 + 17 + 12 - 10 = 42.$$

Problem 3. Peter deleted one digit from a four-digit prime number and obtained 630. What was the prime number?

Result. 6301

Solution. Since the last digit of a (four-digit) prime number cannot be even, the prime number was of the form $630*$. Moreover, the last digit cannot be 5, because the number would be divisible by 5. So the options 1, 3, 7 and 9 are left. But since 630 is divisible by 3, the digits 3 and 9 are impossible. Similarly, 630 is divisible by 7 and so would be 6307. Therefore, the number was 6301.

Problem 4. Star architect Pegi wants to build a very modern pentagonal mansion on a rectangular lot of land having side lengths 35 m and 25 m. The floor area of the mansion fits into the lot as can be seen in the picture:



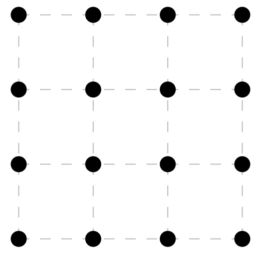
(The dots on the boundary mark 5 m distance.) What fraction of the area of the lot does the floor area of the mansion cover?

Result. $\frac{41}{70}$

Solution. Since we are only interested in the fraction of areas, we can use 5 m as a unit. By summing up the areas of the three right-angled triangles, we obtain the result

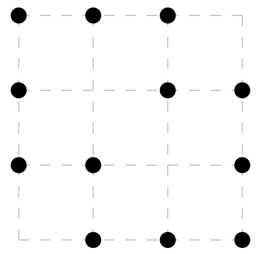
$$1 - \frac{1}{5 \cdot 7}\left(\frac{5 \cdot 3}{2} + \frac{6 \cdot 1}{2} + \frac{4 \cdot 2}{2}\right) = \frac{41}{70}.$$

Problem 5. A square grid of 16 dots, as can be seen in the picture, contains the corners of nine 1×1-squares, four 2×2-squares, and one 3×3-square, for a total of 14 squares whose sides are parallel to the sides of the grid. What is the smallest possible number of dots you can remove so that, after removing those dots, each of the 14 squares is missing at least one corner?



Result. 4

Solution. It is necessary to remove four dots, because the four 1×1-squares in the corners of the given grid do not share any dots in common. Removing two opposite corners of the grid and two centre dots along the other diagonal provides an example to show that this number is sufficient.



Problem 6. Find the digit at the units place of the sum of squares

$$1^2 + 2^2 + 3^2 + \cdots + 2017^2.$$

Result. 5

Solution. The digits in the unit position of square numbers show up periodically having period 10. We have

$$1^2 + 2^2 + 3^2 + \cdots + 10^2 = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385,$$

so the last digit is 5. Consequently, the last digit of $1^2 + 2^2 + \cdots + 2010^2$ is 5 due to $201 \cdot 5 = 1005$. Furthermore, the last digit of the sum $2011^2 + 2012^2 + \cdots + 2017^2$ is 0. Altogether, the desired digit is 5.

Problem 7. Express the quotient

$$\frac{0.\overline{2}}{0.\overline{24}}$$

as a fraction $\dfrac{a}{b}$ in lowest terms with positive integers a and b.

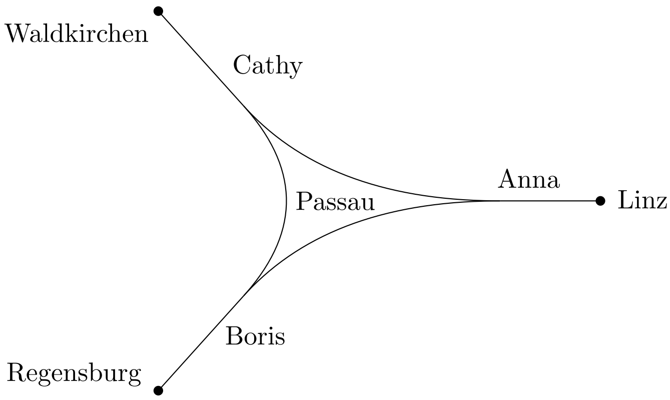
Note: The overline denotes periodic decimal expansion, for example $0.\overline{123} = 0.123123\ldots$.

Result. $\frac{11}{12}$

Solution. The given quotient can be written in the following way:

$$\frac{0.\overline{2}}{0.\overline{24}} = \frac{0.\overline{22}}{0.\overline{24}} = \frac{22 \cdot 0.\overline{01}}{24 \cdot 0.\overline{01}} = \frac{11}{12}.$$

Problem 8. Passau has a railway station in the form of a triangle. Anna, Boris, and Cathy observe the railway traffic in Linz, Regensburg, and Waldkirchen, respectively, on the rails coming from Passau. Anna counts 190, Boris 208 and Cathy 72 incoming and outgoing trains in total. How many trains went from Linz to Regensburg or vice versa if no train starts, ends or reverses its direction in Passau?



Result. 163

Solution. We denote the number of trains between Linz and Waldkirchen by r, the one between Linz and Regensburg by w, and the one between Waldkirchen and Regensburg by l. Anna counts all the trains between Linz and the other two cities. Therefore, we obtain the equation $r + w = 190$. We get the equations $l + w = 208$ and $l + r = 72$ similarly. Adding the first two equations and subtracting the third one gives $2w = 190 + 208 - 72$ which leads to $w = \frac{1}{2} \cdot 326 = 163$.

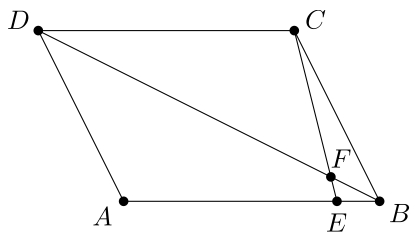
Problem 9. Find all positive integers $x < 10\,000$ such that x is a fourth power of some even integer and one can permute the digits of x in order to obtain a fourth power of some odd integer. The result of permuting may not begin with zero.

Result. 256

Solution. Assume that x is equal to a^4 for some even integer a and that it can be transformed to b^4 for some odd integer b.

Since $10\,000 = 10^4$, both a and b are less than 10. By squaring even one-digit integers and then squaring their last digits, we find out that a^4 always ends with 6. Considering the possible values of b, we see that only $5^4 = 625$ and $9^4 = 6561$ contain the digit 6. However, any permutation of digits of 9^4 would be divisible by 3 (digit sum does not change), so the only possible result would be $6^4 = 1296$, which cannot be permuted to 9^4. For $b^4 = 625$ we easily find $x = a^4 = 256$ as the only result.

Problem 10. In parallelogram $ABCD$ a line through point C meets side AB in point E such that $EB = \frac{1}{5}\,AE$. The line segment CE intersects the diagonal BD in point F. Find the ratio $BF : BD$.



Result. $1 : 7$

Solution. The triangles EBF and CDF are similar with a scale factor $EB : DC = 1 : 6$. As a consequence $BF : FD = 1 : 6$ as well, hence $BF : BD = 1 : 7$.

Problem 11. A big house consists of 100 numbered flats. In every flat, there lives one person or there live two or three persons. The total number of people living in the flats No. 1 to No. 52 is 56 and the total number of people living in the flats No. 51 to No. 100 is 150. How many people live in this house?

Result. 200

Solution. Since the maximum number of people in a flat is three, exactly three people live in each flat from No. 51 to No. 100. Therefore, there are $56 - 2 \cdot 3 = 50$ people living in the first up to the 50th flat in total. This leads to a total number of $50 + 150 = 200$ inhabitants for the entire house.

Problem 12. In the first stage, Nicholas wrote the number 3 with a red pencil and 2 with a green pencil on a sheet of paper. In the following stages, he used the red pencil for writing the sum of the two numbers from the previous stage and the green pencil for their (positive) difference. What number did he write in red in the 2017th stage?

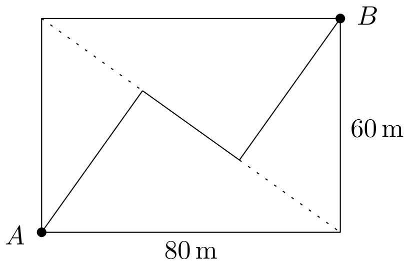
Result. $3 \cdot 2^{1008}$

Solution. It is easy to see that in each stage, the red number is greater than the green one. Assuming that in the n-th stage the numbers R_n, G_n are written with red and green, respectively, then in the stage $n+1$, we have $R_{n+1} = R_n + G_n$ and $G_{n+1} = R_n - G_n$ and in the stage $n+2$,

$$\begin{aligned} R_{n+2} &= R_{n+1} + G_{n+1} = 2R_n, \\ G_{n+2} &= R_{n+1} - G_{n+1} = 2G_n. \end{aligned}$$

Therefore, both the numbers are doubled each two stages. Between the 1st and the 2017th stage this happens 1008 times, therefore the result is $3 \cdot 2^{1008}$.

Problem 13. Little Red Riding Hood finds herself at the entrance of the "Rectangular Forest". Starting at point A, she has to reach point B as fast as possible. One possibility is to walk along the edges of the woods which will be 140 m in total. Of course, she knows that according to the triangle inequality a direct path would be shorter. Unfortunately, there is only a path which has the form of a zigzag with two right-angled turns as can be seen in the picture. If she knew that this way was shorter than 140 m, she would dare to take the shortcut. Find the length (in meters) of the zigzag through the forest!



Result. 124

Solution. By the Pythagorean theorem, the length of the diagonal of the rectangle is $\sqrt{60^2 + 80^2} = 100$. For example by the cathetus theorem, one can determine the shorter segment cut by the altitude to the diagonal as $60^2/100 = 36$. Now we obtain $100 - 36 = 64$ for the longer segment of the diagonal. Using the altitude theorem, the length of the altitude is $\sqrt{36 \cdot 64} = 48$. Altogether, the zigzag path is of length $48 + (64 - 36) + 48 = 124$.

Problem 14. Eight-digit palindromes are numbers of the form $\overline{abcddcba}$ where a, b, c, and d are not necessarily distinct digits. How many eight-digit palindromes have the property that we can delete some digits in such a way that the resulting number would be 2017?

Result. 8

Solution. Since all digits of 2017 are different, all digits a, b, c, d are distinct and we should erase each letter exactly once. Observe that after erasing, either the first or the last digit of 2017 will be equal to a. In the former case we have $a = 2$ and arrive at an analogous problem for a six-digit palindrome with digits 0, 1, 7. In the latter case we have $a = 7$ and arrive at an analogous problem for a six-digit palindrome with digits 2, 0, 1. Again, in both cases we have two choices for the first digit, so we obtain four problems for a four-digit palindrome. Again, each of these problems splits into two problems for a two-digit palindrome, which has a unique solution. Hence, overall we have $2^3 = 8$ options.

Problem 15. For a positive integer n the sum of its digits is denoted by $\mathrm{S}(n)$, and the product of its digits by $\mathrm{P}(n)$. How many positive integers n are there having the property $n = \mathrm{S}(n) + \mathrm{P}(n)$?

Result. 9

Solution. For a positive integer n consisting of only one digit, we always have $\mathrm{S}(n) + \mathrm{P}(n) = 2n > n$. Now we consider positive integers n having more than one digit. Let $m \geq 1$ and $n = a_m 10^m + \cdots + a_0$ be an integer with $0 \leq a_k \leq 9$ for $0 \leq k \leq m$ and $a_m \neq 0$. Then we have

$$\begin{aligned} n - \mathrm{S}(n) - \mathrm{P}(n) &= a_m 10^m + \cdots + a_0 - (a_m + \cdots + a_0) - a_m \cdots a_0 \\ &= (10^m - 1 - a_{m-1} \cdots a_0) a_m + (10^{m-1} - 1) a_{m-1} + \cdots + 9a_1 \\ &\geq (10^m - 1 - 9^m) a_m \\ &\geq 0 \end{aligned}$$

and equality can hold only for $m = 1$. Therefore, for n to satisfy the given condition we must have

$$n = 10a_1 + a_0 = a_1 + a_0 + a_1 a_0$$

which is equivalent to $a_1(9 - a_0) = 0$, that is to $a_0 = 9$. We conclude that exactly the nine numbers 19, 29, 39, 49, 59, 69, 79, 89, and 99 have the desired property.

Problem 16. A factory owner hires 100 employees. Each work team leader earns 5 000 € per month, each labourer 1 000 € per month and each part-time worker earns 50 € per month. In total, the factory owner pays 100 000 € monthly to his employees and there is at least one employee of each type. How many work team leaders are there in his factory?

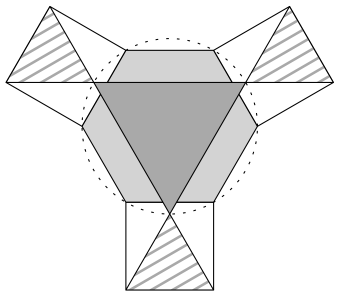
Result. 19

Solution. Denote by x, y, z the number of work team leaders, labourers, and part-time workers, respectively. The conditions from the statement can be rewritten into a system of equations

$$x + y + z = 100 \tag{1}$$
$$5\,000x + 1\,000y + 50z = 100\,000 \tag{2}$$

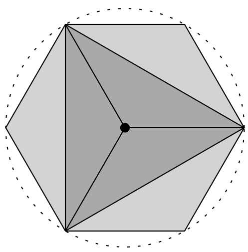
in positive integers. Expressing z from (2) yields $z = 2000 - 20y - 100x$ with the right-hand side divisible by 20, so z can be written as $z = 20k$ for some positive integer k. Therefore, (2) can be simplified to $5x + y + k = 100$. Subtracting (1), wherein $z = 20k$ is used as well, leads to $4x = 19k$. As 4 and 19 are coprime, it follows that x is a multiple of 19. Since x, y, and z are positive, (2) implies $x < 20$. Consequently, $x = 19$ (with $y = 1$ and $z = 80$) is the only solution.

Problem 17. The mosaic below is composed of regular polygons. The hexagon and the dark grey triangle are inscribed in the same circle. If the area of a striped triangle is 17, determine the area of the dark grey triangle.



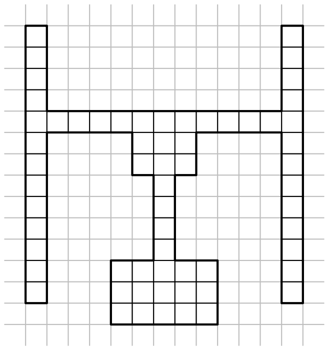
Result. 51

Solution. Firstly, consider the dark grey triangle being rotated by 30° about the center of the circle such that its vertices coincide with vertices of the hexagon. Then it is clear that it covers half of the hexagon.



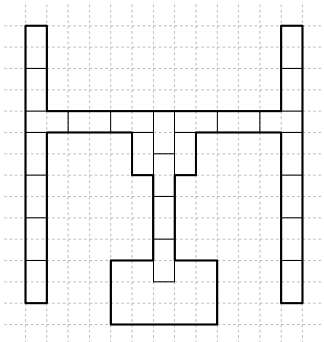
Secondly, observe that a striped triangle is equilateral having the same side length as the hexagon. Therefore, the area of a striped triangle is $\frac{1}{6}$ of the area of the hexagon. Altogether, the area of the dark grey triangle is $3 \cdot 17 = 51$.

Problem 18. In the course of renovating the train station of Passau, special paving for visually handicapped people is being built in. The shape of the paving is shown in the picture below. Unfortunately, only tiles of the size 1×2 are available. In how many ways can the paving be filled with tiles? The tiles are indistinguishable and two pavings are considered to be different if at some spot tiles are in a different position.

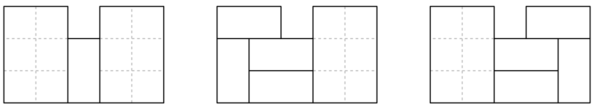


Result. 15

Solution. By starting to place the tiles along the narrow strips, it is obvious that the way of tiling is determined up to the 3×5-area:

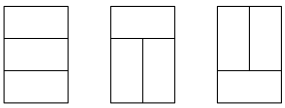


There are three options for the next tile: Either it can be put in the same direction as the last one (case (1) below), which produces two separated 3×2-areas, or it can be put perpendicularly (cases (2) and (3)). In these cases we may carry on tiling in a unique way until all but one 3×2-area remains.



(1) (2) (3)

Each 3×2-area can be tiled in three ways as in the following picture:



Therefore, there are $3 \cdot 3 = 9$ ways to finish the tiling in the case (1) and 3 ways in both cases (2) and (3). Altogether, we get $9 + 3 + 3 = 15$ possibilities.

Problem 19. Michael picked a positive integer n. Then he chose a (positive) divisor of n, multiplied it by 4, and subtracted this result from n, getting 2017. Find all numbers that Michael could have picked.

Result. 2021, 10085

Solution. For the chosen divisor d of n, we have $n = kd$ for some integer k. Now the equation reads

$$kd - 4d = (k - 4)d = 2017.$$

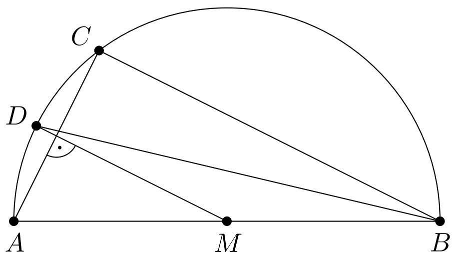
Since 2017 is a prime number, we obtain either $d = 1$ or $d = 2017$. In the first case, we get $n = k = 2017 + 4 = 2021$. In the second case, we obtain $k = 5$ and hence $n = 2017 \cdot 5 = 10085$.

Problem 20. Grisha and Vanechka are very good friends, so whenever they sit next to each other they begin chatting. Five students (including Grisha and Vanechka) want to have a constructive discussion, so they want to take five chairs around a round table in such a way that Grisha and Vanechka are not neighbours. In how many ways can they do it? Arrangements that differ by rotations are different.

Result. 60

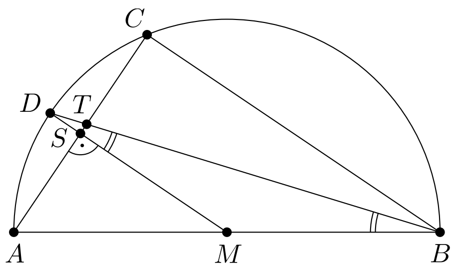
Solution. Once Grisha has taken one of the five places, there are only 2 positions left to seat Vanechka in order not to be next to Grisha. This yields $5 \cdot 2$ possibilities. For each of them, the three remaining students can be seated in 3! ways, which leads to $5 \cdot 2 \cdot 3! = 60$ ways in total.

Problem 21. In the figure, AB is a diameter of a circle with center M. The two points D and C are on the circle in such a way that $AC \perp DM$ and $\angle MAC = 56°$. Find the size of the acute angle between the lines AC and BD in degrees.



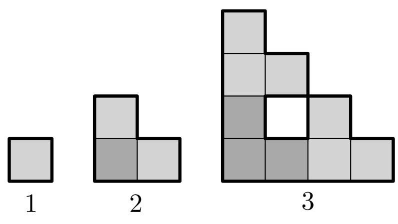
Result. 73°

Solution. Denote the point of intersection of the lines AC and DM by S and the point of intersection of the lines AC and BD by T.



From $\angle MAC = 56°$ and the given right angle at S, we get $\angle SMA = 34°$ using the sum of angles of triangle AMS. Further, $\angle BMD = 180° - \angle SMA = 146°$. As both MD and MB are radii of the circle, $\triangle BDM$ is isosceles ($\angle MBD = \angle MDB$) and using the sum of angles again results in $\angle MDB = 17°$. Our goal is to find $\angle CTB$, which equals $\angle STD$; however, this can be computed using the sum of angles of the right triangle DST with the knowledge of $\angle SDT = \angle MDB$. The result is $\angle CTB = 73°$.

Problem 22. An object is built from unit squares by successively composing its copies as indicated in the figure below. What is the length of the thick boundary of the object in the stage 6?



Result. 488

Solution. Let f_n denote the length of the thick boundary in the n-th stage. Note that in each stage three congruent figures from the previous stage are glued together along two unit segments which are no longer parts of the boundary. This observation yields $f_{n+1} = 3 \cdot f_n - 2 \cdot 2$ for all $n \geq 1$ which combined with $f_1 = 4$ gives $f_6 = 488$ after direct computation.

Problem 23. All the 2017 seats around a very large round table are occupied by superheroes and villains. The superheroes always tell the truth, whereas the villains always lie. Each person sitting at the table reported that he or she was sitting between a superhero and a villain. For unknown reasons, exactly one superhero made a mistake. How many superheroes are there?

Result. 1345

Solution. Firstly, observe that there are no villains sitting next to each other: if that was the case, another villain would have to sit on either side, next to this one another one etc., implying that no superheroes are present, which is not true because the statement says that there is at least one. Assuming first that all superheros tell the truth, every superhero sits between another superhero and a villain, so the whole party at the table can be divided into segments of the form superhero-superhero-villain. The lying superhero can now be either seated between two superheroes, or inserted together with one more villain between a superhero and a villain. The remainder of the division of the total number of people by three would be 1 in the former case and 2 in the latter case. Since 2017 gives remainder 1 when divided by 3, the former case applies and we conclude that there are $\frac{2}{3} \cdot 2016 + 1 = 1345$ superheroes at the table.

Problem 24. Find all positive real numbers x such that

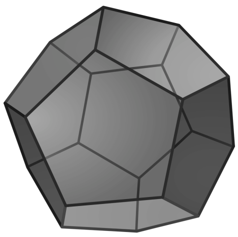
$$x^{2017x} = (2017x)^x.$$

Result. $\sqrt[2016]{2017}$

Solution. Since x is positive, we may raise both sides to the power $1/x$ and obtain $x^{2017} = 2017x$ or $x^{2016} = 2017$. The solution is $x = \sqrt[2016]{2017}$.

Problem 25. Leo wants to colour the edges of a regular dodecahedron in a special way: He chooses a felt pen of one colour, starts at one vertex of the dodecahedron and moves along connected edges without lifting his pen and without colouring any edge twice until he wants to or is forced to stop. Then he takes another felt pen and begins colouring some connected uncoloured edges. He continues this procedure using one colour at a time until every edge of the dodecahedron is coloured exactly once. What is the minimum number of colours which can be used?

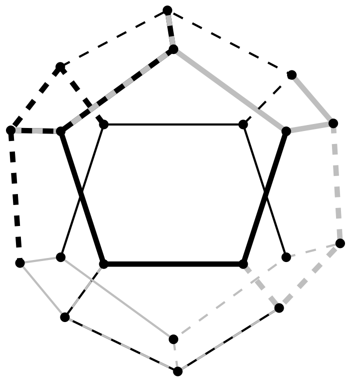
Note: Regular dodecahedron is a regular solid with twelve pentagonal faces as shown in the picture:



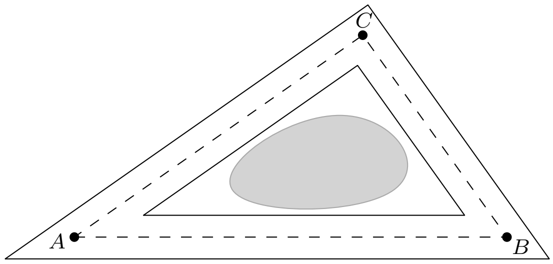
Result. 10

Solution. A regular dodecahedron has 20 vertices and at every vertex 3 edges meet. Now consider only the graph model consisting of nodes (vertices) and edges of the dodecahedron. After each step of colouring a path of connected edges with one colour, remove those coloured edges from the model. If a closed path is taken out, at each node either 0 or 2 edges are removed. If the beginning node b and the ending node e of the removed path are different, exactly one edge is taken out at the nodes b and e whereas at the remaining nodes 0 or 2 edges are eliminated. Therefore, each node has to be a starting/ending node of at least one path and so there must be at least 10 paths (colours).

The following picture shows that a colouring using 10 colours exists:

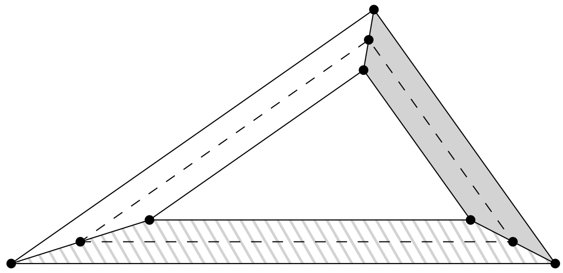


Problem 26. Landscape gardener Joe designed a new gravel path around a lake. The triangle ABC marks the middle of the path, its side lengths are $a = 80\,\mathrm{m}$, $b = 100\,\mathrm{m}$ and $c = 120\,\mathrm{m}$. The borders of the path have distance $1\,\mathrm{m}$ from this triangle, as can be seen in the following picture. How many m^3 of fine loose gravel must Joe order if the covering with gravel should be of height $4\,\mathrm{cm}$ in average?



Result. 24

Solution. The area of the path can be subdivided into three trapezoids having height 2 and midline a, b, and c, respectively.



Since the area of a trapezoid can also be computed by multiplying its height by its midline, we get

$$2 \cdot (80 + 100 + 120) = 2 \cdot 300 = 600$$

as the result for the area of the path in m^2. The amount of gravel to be ordered can now be computed as $600\,\mathrm{m}^2 \cdot 0.04\,\mathrm{m} = 24\,\mathrm{m}^3$. So the answer is 24.

Problem 27. Find all four-digit squares of integers such that the first two digits are equal and the last two digits are also equal.

Result. 7744

Solution. Let N be the number in question and denote by x and y its first and last digit, respectively. Then we have

$$N = 1000x + 100x + 10y + y = 11(100x + y),$$

so N is divisible by 11. But N being a square number, it also has to be divisible by 11^2. Thus we get $N = 11^2k^2$ for some positive integer k and $100x + y = 11k^2$. Since the left-hand side of this equality is a three-digit integer $\overline{x0y}$ with the tens digit being zero, k^2 has to be a two-digit square number whose digits sum up to 10. The only number of this kind is $8^2 = 64$. Therefore, we have $100x + y = 11 \cdot 8^2$ and obtain $N = 11^2 \cdot 8^2 = 88^2 = 7744$.

Problem 28. At a fun fair there is a lottery drawing with the following rules: a participant has to choose one of four indistinguishable boxes and afterwards he draws one ball out of the chosen box. If this ball is white, the participant is a winner, if it is black, he loses. For example, if the distribution of the balls in the four boxes is

$$(6,6),\ (5,3),\ (4,0),\ (3,5),$$

where each pair (w, b) represents a box with w white and b black balls inside, then a participant wins with a probability of $\frac{5}{8}$. Every 1000^{th} participant receives a super-joker: he or she is allowed to redistribute all the balls among the boxes arbitrarily, putting at least one ball in each box. Then the boxes are mixed again and the participant can choose a box and draw one ball. Johanna is a lucky girl and has won the super-joker. What is the largest probability for a win she can achieve by using an appropriate distribution of the balls given in the example?

Result. $\frac{51}{58}$

Solution. If Johanna puts exactly one white ball each in three of the four boxes, she wins in any case if she chooses one of these boxes. Given the distribution $(1,0)$, $(1,0)$, $(1,0)$, $(15,14)$ for (w,b), she wins with probability $\frac{1}{4}(1+1+1+\frac{15}{29}) = \frac{51}{58}$.

It is easy to see that every other distribution leads to a smaller probability for a win: the probability of a loss is the sum of probabilities of choosing each single black ball and for a single ball, the probability to be chosen is clearly minimal if it is in a box with as many balls as possible.

Problem 29. A bus, a truck, and a motorcycle move at constant speeds and pass a stationary observer in this order at equal time intervals. They pass another observer farther down the road at the same equal time intervals, but in different order: this time the order is bus, motorcycle, truck. Find the speed of the bus in km/h, if the speed of the truck is 60 km/h, and the speed of the motorcycle is 120 km/h.

Result. 80

Solution. Let t be the common time interval between the moments the vehicles pass the observers. The motorcycle reaches the first observer t hours after the truck and reaches the second observer t hours before the truck. Therefore, the motorcycle covers the distance between the two observers $2t$ hours faster than the truck does. The motorcycle is travelling twice as fast as the truck, so it covers the distance between the two observers in half the time. Consequently, the truck must take $4t$ hours to cover the distance between the observers while the motorcycle takes $2t$ hours.

Now the bus passes the first observer t hours ahead of the truck and it passes the second observer $2t$ hours ahead of the truck. The truck takes $4t$ hours to go between the observers, so the bus takes $3t$ hours to go the same distance. Consequently, the bus is going $\frac{4}{3}$ the speed of the truck and this is 80 km/h.

Problem 30. Find all ways to fill all gaps in the statement below with positive integers in order to make the statement true: "In this statement, $\boxed{}$% of digits is greater than 4, $\boxed{}$% of digits is less than 5, and $\boxed{}$% of digits is equal to either 4 or 5".

Result. 50, 50, 60

Solution. Clearly, there cannot be more than ten digits in the statement. Therefore, using the digits we already know, we observe that the first two gaps should be filled with at least 20 and the third one with at least 40. Thus there are altogether ten digits and all numbers in the gaps should end with 0, so we can denote them by $\overline{a0}$, $\overline{b0}$, and $\overline{c0}$, respectively, with digits a, b, and c such that $a + b = 10$.

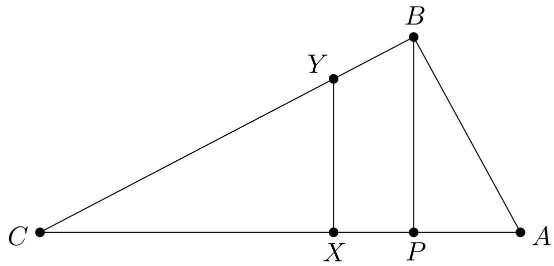
Because there are at least two digits greater than 4 and at least five digits less or equal 4, we can conclude that $5 \geq a \geq 2$. Moreover, at least one of the digits a and b is greater than 4, so we get $5 \geq a \geq 3$.

Having already four digits equal to either 4 or to 5 means $c \geq 4$. Since the definition of $\overline{c0}\,\%$ excludes the case $c = 4$, we must have $c \geq 5$ which yields $5 \geq a \geq 4$. In case $a = 4$, we get $b = 6$, but then for every $c \geq 5$, it is not possible to obtain a true statement as requested. Now let $a = 5$. This leads to $b = 5$ and the statement is true for $c = 6$.

Problem 31. Paul's cattle grazes on a triangular meadow ABC. Since his spotted cows and those without spots did not go well together, Paul built a twenty-meter long fence perpendicular to the side AC, starting at point P on this side and ending in B, dividing the meadow into two right-angled triangles. However, this was soon met with protests from the spotted cows (grazing in the part with point A), which pointed out that $AP : PC = 2 : 7$ and demanded a fairer division. Consequently, Paul replaced the fence with a new one, parallel to the old one and with ends on the borders of the meadow, but dividing the meadow into two parts of equal area. What was the length of the new fence in meters?

Result. $30\sqrt{\frac{2}{7}}$

Solution. Since triangle PBC has greater area than ABP, the endpoint of the new fence on side AC (denoted by X) lies between C and P, and the other endpoint (denoted by Y) lies on side BC. Triangles PBC and XYC are similar, so $XY : XC = PB : PC$.



Further, as the area of triangle XYC is one half of the area of triangle ABC,

$$\frac{1}{2} \cdot XY \cdot XC = \frac{1}{2} \cdot \frac{1}{2} \cdot PB \cdot AC$$

or

$$XY^2 = \frac{1}{2} \cdot \frac{XY}{XC} \cdot PB \cdot AC = \frac{1}{2} \cdot PB^2 \cdot \frac{AP + PC}{PC},$$

therefore

$$XY = PB \cdot \sqrt{\frac{1}{2}\left(1 + \frac{AP}{PC}\right)} = 30\sqrt{\frac{2}{7}}.$$

Problem 32. There were five (not necessarily distinct) real numbers written on a blackboard. For every pair of these numbers, Wendy calculated their sum and wrote the ten results

$$1,\ 2,\ 3,\ 5,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10$$

on the blackboard, erasing the initial numbers. Determine all possible values of the product of the erased numbers.

Result. -144

Solution. Denote the initial numbers by $a \le b \le c \le d \le e$. Among the ten calculated sums, the smallest is $a+b=1$, the second smallest is $a+c=2$, the largest is $d+e=10$, and the second largest is $c+e=9$. Since all ten sums add up to

$$4(a+b+c+d+e) = 1+2+3+5+5+6+7+8+9+10 = 56,$$

we get $a+b+c+d+e=14$ and $c = 14-1-10 = 3$. It follows that $a = 2-c = -1$, $b = 1-a = 2$, $e = 9-c = 6$, and $d = 10-e = 4$. For these values we can easily check that the remaining six sums exactly fit into the list of numbers on the blackboard. Therefore, the desired product is $-1\cdot 2\cdot 3\cdot 4\cdot 6 = -144$.

Problem 33. Write 333 as the sum of squares of (arbitrarily many) distinct positive odd integers.

Result. $3^2+5^2+7^2+9^2+13^2$

Solution. Since $17^2 = 289 < 333 < 361 = 19^2$, only the numbers $1^2, 3^2, \ldots, 17^2$ (altogether nine distinct numbers) may appear as summands. Further, whenever an odd integer is squared, the result gives remainder 1 when divided by 8—since 333 has remainder 5 after dividing by 8, the number of summands in the desired sum has to be five.

Since $1^2+3^2+5^2+7^2+17^2 > 333$, 17^2 cannot appear in the sum. Let us take a look at the leftover summands modulo 5: Two of them are divisible by 5 (5^2, 15^2), three of them give remainder 1 (1^2, 9^2, 11^2) and three give remainder -1 (3^2, 7^2, 13^2). Since 333 gives remainder 3 (or -2), there are two cases to consider. Firstly, we may sum up all the numbers with remainder 0 or 1; however, this turns out to exceed 333. Secondly, to achieve -2, we have to sum all the numbers with remainder -1, one with remainder 0 and one with 1. It is easy to see that the results containing 11^2 or 15^2 are too large, and out of the two remaining possibilities, only $3^2+5^2+7^2+9^2+13^2$ is equal to 333.

Problem 34. Ellen picked three real numbers a, b, c and defined the operation $\odot$ as $x \odot y = ax + by + cxy$. As an exercise, she computed that $1 \odot 2 = 3$ and $2 \odot 3 = 4$. After further investigation, she noticed that there was a non-zero real number u such that $z \odot u = z$ for every real z. What was the value of u?

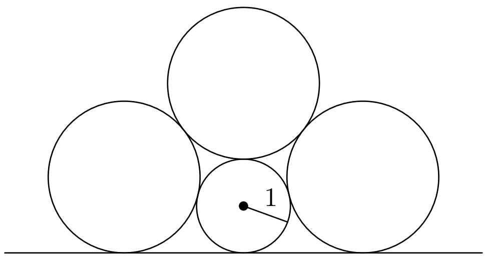
Result. 4

Solution. From $0 = 0 \odot u = bu$ we get $b = 0$ (u is non-zero). The equations in the statement now may be rewritten as

$$\begin{aligned} a + 2c &= 3, \\ 2a + 6c &= 4 \end{aligned}$$

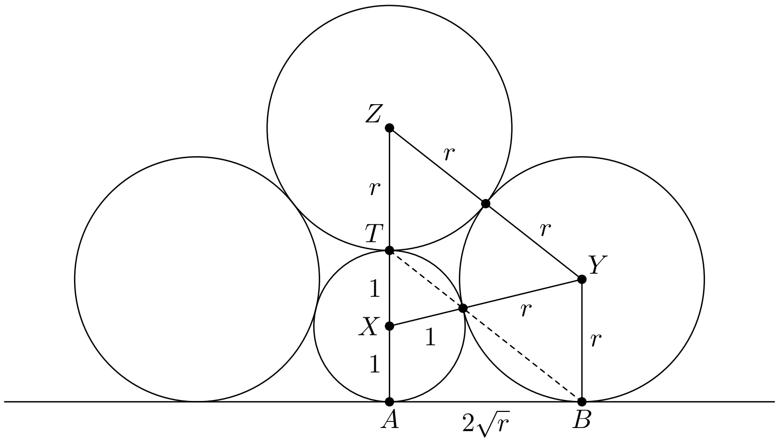
with solution $a = 5$, $c = -1$. Finally $1 = 1 \odot u = 5 - u$ yields $u = 4$.

Problem 35. Three circles of radius r and one circle of radius 1 touch each other and a straight line as shown in the picture. Find r.



Result. $\frac{1+\sqrt{5}}{2}$

Solution. Denote certain points of contact by A, B, and T and centers of circles by X, Y, and Z as shown in the following picture:



Applying the Pythagorean theorem, we get

$$AB^2 = XY^2 - (BY - AX)^2 = (r+1)^2 - (r-1)^2 = 4r.$$

As $BY \parallel TZ$ and $BY = TZ = r$, the quadrilateral $BYZT$ is a parallelogram and $BT = YZ = 2r$. Since the Pythagorean theorem in triangle ABT yields $AB^2 + AT^2 = BT^2$, we now obtain $4r + 2^2 = (2r)^2$, which reduces to

$$r^2 - r - 1 = 0.$$

The only positive solution of this quadratic equation is $r = \frac{1+\sqrt{5}}{2}$.

Problem 36. On an old concrete wall, someone had sprayed five (not necessarily distinct) real numbers, the sum of which was 20. For each pair of these numbers, Harry computed their sum and rounded it down (i.e. took the greatest non-exceeding nearest integer), thus obtaining ten integers. Finally, he added together all these new numbers. What was the smallest possible value of the sum that Harry could get?

Result. 72

Solution. Let $a_1, \ldots, a_5$ be the numbers on the wall. We are seeking the smallest possible value of

$$W = \sum_{1 \le i < j \le 5} \lfloor a_i + a_j \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. We can rewrite this expression as

$$\sum_{1 \le i < j \le 5} \lfloor a_i + a_j \rfloor = \sum_{1 \le i < j \le 5} (a_i + a_j) - \sum_{1 \le i < j \le 5} \{a_i + a_j\} = 80 - \sum_{1 \le i < j \le 5} \{a_i + a_j\},$$

where $\{x\}$ stands for the fractional part of x (i.e. $\{x\} = x - \lfloor x \rfloor$). Therefore our goal is to maximize the sum

$$S = \sum_{1 \le i < j \le 5} \{a_i + a_j\}.$$

This can be further split into two sums

$$\sum_{i=1}^{5} \{a_i + a_{i+1}\} + \sum_{i=1}^{5} \{a_i + a_{i+2}\}$$

(letting $a_6 = a_1$ and $a_7 = a_2$). Now the equalities

$$S_1 = \sum_{i=1}^{5} \{a_i + a_{i+1}\} = 40 - \sum_{i=1}^{5} \lfloor a_i + a_{i+1} \rfloor,$$
$$S_2 = \sum_{i=1}^{5} \{a_i + a_{i+2}\} = 40 - \sum_{i=1}^{5} \lfloor a_i + a_{i+2} \rfloor$$

imply that both S_1, S_2 are integers. Further, they are both sums of five numbers less than 1, and so both are at most 4; consequently, $S \le 8$. For the original sum we thus obtain $W \ge 72$ and the equality is achievable by letting all the fractional parts of $a_1, \ldots, a_5$ be 0.4, for example $a_1 = 2.4$, $a_2, \ldots, a_5 = 4.4$.

Problem 37. For a composite positive integer n, let $\xi(n)$ denote the sum of the least three divisors of n, and $\vartheta(n)$ the sum of the greatest two divisors of n. Find all composite numbers n such that $\vartheta(n) = (\xi(n))^4$.

Note: By a divisor we mean a positive, not necessarily proper divisor.

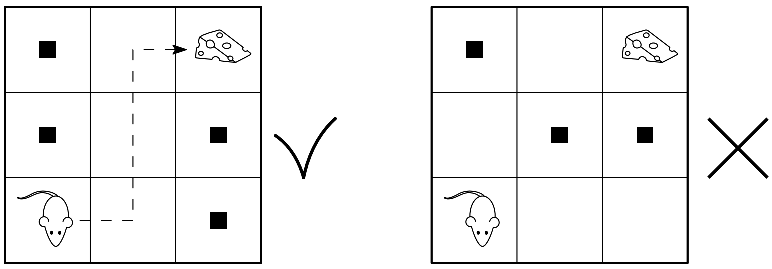
Result. 864

Solution. The smallest divisor of n is 1. Let the second and third smallest divisors of n be p, q respectively (so p is a prime and q is either a prime or $q = p^2$). Then $\xi(n) = 1 + p + q$ and $\vartheta(n) = n + n/p$. So after multiplying by p, the condition from the statement can be rewritten as

$$n(p + 1) = p(1 + p + q)^4.$$

The right-hand side is divisible by $p + 1$ and since p and $p + 1$ are coprime, $p + 1 \mid (1 + p + q)^4$. If both p, q were odd, $p + 1$ would be even and $(1 + p + q)^4$ odd, which is not possible, therefore $p = 2$ (2 has to be a divisor of n and p is the smallest prime divisor). Now expanding $(1 + p + q)^4 = (3 + q)^4$ via the binomial formula and omitting the terms divisible by 3 shows that $3 \mid q^4$, therefore $3 \mid q$. The option $q = p^2$ being clearly impossible, we conclude that q is a prime and so $q = 3$. Finally, $n \cdot 3 = 2 \cdot 6^4$, so $n = 2^5 \cdot 3^3 = 864$.

Problem 38. On a 3×3 board, a mouse sits in the bottom left square. Its goal is to get to a piece of cheese in the top right square using only moves between adjacent squares. In how many ways can we fill some (possibly none) of the free squares with obstacles so that the mouse is still able to reach the cheese?



Result. 51

Solution. Firstly, let us consider the case when we fill the centre square. In such a case the mouse can reach the cheese only by moving along one of the "border corridors", so we may place further obstacles to only one of these corridors. Each corridor consists of three squares, thus there are $2^3 = 8$ ways to fill one of them and $8 + 8 - 1 = 15$ ways to leave at least one corridor free (the -1 is due to the situation with both corridors free being counted twice).

Let us now focus on the case with the centre square free. The path to cheese exists if and only if at least one of the squares adjacent to the mouse and at least one adjacent to the cheese is free. This gives three options for the two squares adjacent to the mouse and three options for the two squares adjacent to the cheese (i.e. at most one filled square) and two options for each of the remaining two corners, so the number of placements plausible for the mouse in this case is $3 \cdot 3 \cdot 2 \cdot 2 = 36$.

We conclude that there are $15 + 36 = 51$ ways in total.

Problem 39. Among all the pairs (x, y) of real numbers satisfying

$$x^2y^2 + 6x^2y + 10x^2 + y^2 + 6y = 42,$$

let (x_0, y_0) be the one with minimal x_0. Find y_0.

Result. -3

Solution. Firstly, we observe that if we add 10 to both sides, the binomial $x^2 + 1$ can be factored out from the left-hand side, so

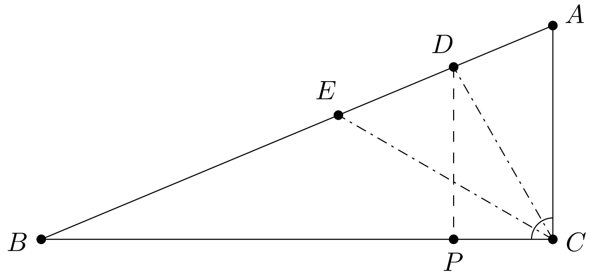
$$(x^2 + 1)(y^2 + 6y + 10) = 52.$$

The expression on the left-hand side can be viewed as a product of two quadratic functions $f(x) = x^2 + 1$ and $g(y) = y^2 + 6y + 10$. Now since the graphs of quadratic functions are symmetric and in this case, both functions are first decreasing and then increasing, the minimal value x_0 gives the maximal value of f, which in turn forces $g(y_0)$ to be minimal possible (the product $f(x)g(y)$ is a positive constant). So it suffices to find where $g(y) = (y + 3)^2 + 1$ attains its minimum and we conclude that the answer is $y_0 = -3$.

Problem 40. Let $\triangle ABC$ be a right-angled triangle with right angle at C. Let D and E be points on AB with D between A and E such that CD and CE trisect $\angle ACB$. If $DE : BE = 8 : 15$, find $AC : BC$.

Result. $\frac{4\sqrt{3}}{11}$

Solution. Let P be a point on BC such that $DP \parallel AC$. As the triangles ABC and DBP are similar, $AC : BC = DP : BP$.



Since CE is the angle bisector in triangle BCD, the Angle bisector theorem implies $CD : CB = ED : EB = 8 : 15$. Further, $DP = CD \cdot \sin 60^\circ$ and $BP = CB - CP = CB - CD \cdot \cos 60^\circ$, therefore

$$\frac{DP}{BP} = \frac{\frac{\sqrt{3}}{2}CD}{\frac{15}{8}CD - \frac{1}{2}CD} = \frac{4\sqrt{3}}{11}.$$

Problem 41. Mike plays the following game: His task is to find an integer between 1 and N (inclusive). In each turn, he picks an integer from this interval; if it is the correct one, the game ends, otherwise he is told if his choice was too large or too small. However, if the picked number is too large, he has to pay 1 €, and if it is too small, he pays 2 € (and he pays nothing if his choice is correct). What is the largest integer N such that Mike can always finish the game provided that he can spend at most 10 €?

Result. 232

Solution. Denote by N_k the maximal N such that with k Euros Mike can always find the integer from the range $1, \dots, N$ (or, equivalently, any set of integers of length N); our goal is to compute N_{10}. Clearly $N_0 = 1$ and $N_1 = 2$. Let us show that the numbers satisfy the recurrence relation

$$N_{k+2} = N_{k+1} + N_k + 1.$$

Indeed, if Mike has $k+2$ euros and picks the number $N_{k+1} + 1$, then either it is the correct one, or it is too large (in which situation he proceeds with $k+1$ money and a set of length N_{k+1}) or too small (leading to k money and N_k choices). This shows $N_{k+2} \geq N_{k+1} + N_k + 1$. On the other hand, if there are more than $N_{k+1} + N_k + 1$ numbers to choose from, then selecting a number greater than $N_{k+1} + 1$ may lead to more than N_{k+1} choices but only $k+1$ money left (too large choice) and similarly, selecting a number less than $N_{k+1} + 2$ may result in more than N_k choices but only k money left (too small choice). Thus the recurrence relation is proved.

The number $N_{10} = 232$ now may be computed via the recurrence in a straightforward way.

Problem 42. Positive integers a, b, c satisfy $a \geq b \geq c$ and

$$a + b + c + 2ab + 2bc + 2ca + 4abc = 2017.$$

Find all possible values of a.

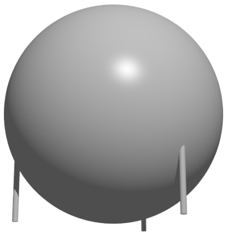
Result. 134

Solution. Observe that if we multiply the left-hand side by 2 and add 1, we get

$$1 + 2a + 2b + 2c + 4ab + 4bc + 4ca + 8abc = (2a+1)(2b+1)(2c+1).$$

The same operation applied on the right-hand side yields $2 \cdot 2017 + 1 = 4035$. Factorising into primes gives $4035 = 3 \cdot 5 \cdot 269$, and since $a \geq b \geq c \geq 1$, the equalities $2a + 1 = 269$, $2b + 1 = 5$, and $2c + 1 = 3$ must hold. Therefore the result is $a = 134$.

Problem 43. An alien spacecraft has the shape of a perfect ball of radius R, supported by three parallel vertical straight legs of length 1 af (alien fathom) and negligible width. The lower ends of the legs form an equilateral triangle of side length 9 af and when the spacecraft rests on a flat surface, the lowest point of the ball precisely touches the surface. What is the radius R (in af)?

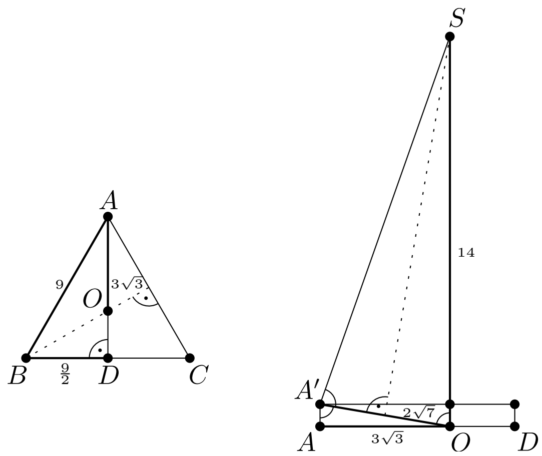


Result. 14

Solution. Denote lower ends of the legs by A, B, C and their upper ends by A', B', C', respectively. Let O be the center of triangle ABC (i.e. the point of tangency the ball to the ground), S be the center of the spacecraft and $D = AO \cap BC$. We have $AO = AB \cdot \frac{\sqrt{3}}{3} = 3\sqrt{3}\,\text{af}$, $AA' = 1\,\text{af}$ and $SO = SA' = R$. Pythagorean theorem yields

$$(SO - AA')^2 + AO^2 = SA'^2 \quad \text{or} \quad (R-1)^2 + 27 = R^2,$$

from which we get $R = 14\,\text{af}$.



Problem 44. Four brothers, Allan, Bert, Charlie, and Daniel, had collected a large amount of hazelnuts in a forest. At night, Allan woke up with the irrestible urge to eat, so he decided to eat some of them. After counting the hazelnuts, he found out that if one hazelnut was removed, the rest could have been divided into four equal parts, so he discarded the extra hazelnut and ate his share (one fourth of the rest); then he returned to bed. Later that night, Bert woke up because of hunger, too, and found out that there was again an extra nut, which he removed and ate one fourth of the remaining nuts. Until morning, exactly the same happened to Charile and Daniel. When all brothers met in the morning, they found out that the number of the remaining nuts was still such that it was divisible by four after removing one nut. What was the minimal number of hazelnuts the brothers could have collected?

Result. 1021

Solution. Let us add three additional, "fake" nuts at the beginning of the night. Then the number of all nuts is a multiple of four. It is enough to observe that after Allan's supper the number of the remaining nuts in the bag is also a multiple of four and there are still the three fake nuts among them. By repeating this reasoning we get that the number of nuts with the fake ones included is equal to 4^5k for some positive integer k, so the actual number of nuts is $4^5k - 3$. Plugging $k = 1$ gives the desired minimum $4^5 - 3 = 1021$.

Problem 45. We call a positive integer *convenient* if all its prime divisors are among 2, 3, and 7. How many convenient numbers are among $1000, 1001, \ldots, 2000$?

Result. 19

Solution. We begin with a general observation: If x is a real number with $x > 1/2$, then there is exactly one (integer) power of 2 in the half-open interval $[x, 2x)$.

Further, call a positive integer *strongly convenient* if all its prime divisors are among 2 and 3. The number of strongly convenient numbers in the interval $[x, 2x)$ can be computed as follows: Firstly, there is exactly one power of two in the interval $[x, 2x)$. Further, in the interval $[x, 2x)$ there is at most one strongly convenient number (say c_1) divisible by 3, but not by 3^2, for $c_1/3$ is the only power of two in the interval $[x/3, 2x/3)$, provided that $x > 1/6$. This way we proceed and find a strongly convenient number c_2 divisible by 3^2 but not by 3^3 etc., until $[x/3^k, 2x/3^k) \cap \mathbb{N} = \emptyset$, i.e., $x < 3^{-k}/2$. We infer that the number of strongly convenient numbers in the interval $[x, 2x)$ is $1+$ the greatest k satisfying $3^k < 2x$; denote this number by $\ell_3(x)$.

To obtain the number of convenient numbers in the interval $[x, 2x)$, we employ a similar technique: The sought number is the sum of numbers of strongly convenient numbers in the intervals $[x, 2x)$, $[x/7, 2x/7)$, $[x/7^2, 2x/7^2)$ etc. Thus the quantity can be computed as $\ell_3(x) + \ell_3(x/7) + \ell_3(x/7^2) + \ldots$, the sum having $\ell_7(x)$ terms, with the symbol ℓ_7 defined in a similar fashion as ℓ_3.

Finally, as 2000 is not a convenient number, our task is to compute

$$\ell_3(1000) + \ell_3(1000/7) + \ell_3(1000/49) + \ell_3(1000/343) = 7 + 6 + 4 + 2 = 19,$$

which is the desired quantity.

Problem 46. Emperor Decimus has banned using the digit 0 (introduced by his predecessor Nullus) and ordered the digit D, representing the quantity 10, to be used instead, thereby founding the *Decimus notation.* Each positive integer still has a unique representation, e.g.

$$3\mathrm{DD}6 = 3 \cdot 1000 + 10 \cdot 100 + 10 \cdot 10 + 6 \cdot 1 = 4106.$$

To smoothen the transfer to the new system, a list of all integers from 1 to DDD (inclusive) has been written down. How many occurrences of the new digit D are there in the list? Multiple occurrences in a single number are counted with multiplicity, so e.g. DD is counted as two D's.

Result. 321

Solution. Observe that all k-digit numbers (in Decimus notation) are represented exactly by the strings

$$\underbrace{1\ldots1}_{k}, \ldots, \underbrace{\mathrm{D}\ldots\mathrm{D}}_{k}.$$

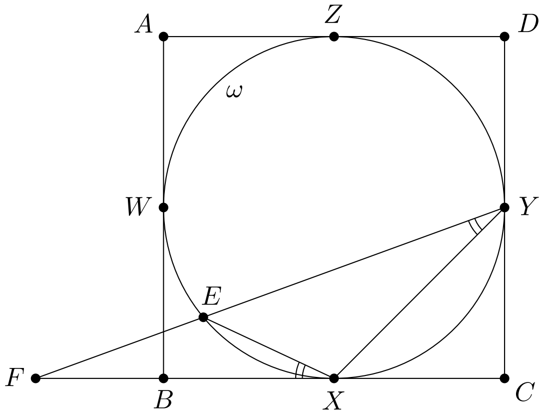
Therefore, to count the total number of D's occurring in the k-digit numbers, we can group together the numbers having D on the first position, the second etc. up to the k-th position. There are exactly 10^{k-1} such numbers for each position—we have to fill the remaining $k-1$ positions with symbols $1, \ldots, 9, \mathrm{D}$. Since for each number we count all occurrences of D, we may distribute the ones with multiple occurrences into all corresponding groups and the result stays correct, so there are $k \cdot 10^{k-1}$ digits D among the k-digit numbers.

It is easy to see that only numbers consisting of one, two, or three digits appear in the list $1, \ldots, \mathrm{DDD}$. We conclude that among them the digit D appears $1 \cdot 10^0 + 2 \cdot 10^1 + 3 \cdot 10^2 = 321$ times.

Problem 47. Square $ABCD$ has inscribed circle ω, which touches the square at points W, X, Y, Z lying on sides AB, BC, CD, DA respectively. Let E be an inner point of the (shorter) arc of ω between W and X and F the intersection of BC and EY. Provided that $EF = 5$ and $EY = 7$, determine the area of triangle FYC.

Result. 21

Solution. By the (limit case of the) Inscribed angle theorem, $\angle EXF = \angle EYX$. Consequently, triangles FEX and FYX are similar (they also share the angle at F), therefore $EF : XF = XF : YF$ or $XF^2 = EF \cdot YF = 5 \cdot 12 = 60$ (a fact also known as the Power of a point theorem).



Let $t = \frac{1}{2}AB$ be one half of the side length of the square. Then the Pythagorean theorem gives

$$YF^2 = t^2 + (t + XF)^2 = 2t^2 + 2t \cdot XF + XF^2 = 2t(t + XF) + XF^2,$$

hence the sought area of triangle can be computed as

$$\tfrac{1}{2} \cdot YC \cdot CF = \tfrac{1}{2} \cdot t \cdot (t + XF) = \tfrac{1}{4}(YF^2 - XF^2) = 21.$$

Problem 48. In the cryptogram

$$WE \cdot LIKE = NABOJ$$

different letters stand for different digits. Furthermore, we are given $\mathrm{S}(WE) = 11$, $\mathrm{S}(LIKE) = 23$, and $\mathrm{S}(NABOJ) = 19$, where $\mathrm{S}(n)$ denotes the sum of digits of a positive integer n. None of the three numbers may start with zero. Find the five-digit number $NABOJ$.

Result. 60 724

Solution. Since in the given cryptogram ten different letters show up, all ten digits $0, 1, \ldots, 9$ are involved. The sum of all digits is 45. By adding up the three given digital sums, we get $11 + 23 + 19 = 53$, which is equal to $45 + E$. This yields $E = 8$, and by squaring it, $J = 4$. Using $\mathrm{S}(WE) = 11$, we also get $W = 3$.

Next, observe that

$$LIKE < \frac{100000}{38} < 2636.$$

Therefore, we have $L = 1$ or $L = 2$. In case $L = 2$, the digital sum of $LIKE$ leads to $I + K = 13$, which means that (I, K) or (K, I), respectively, can be one of the pairs $(4, 9)$, $(5, 8)$, or $(6, 7)$. But none of them is possible any more, since the digits 4 and 8 are already taken and in case of $(6, 7)$, the factor $LIKE$ would exceed the upper bound above. From that we conclude $L = 1$. Then the digital sum of $LIKE$ leads to $I + K = 14$, which restricts the options for (I, K) or (K, I), respectively, to the pair $(5, 9)$. An easy calculation shows that only $(I, K) = (5, 9)$ solves the cryptogram. From $38 \cdot 1598 = 60\,724$ we obtain the desired number $NABOJ = 60\,724$.

Problem 49. Find all positive integers n with the property that the sum of all non-trivial divisors is 63.

Note: A non-trivial divisor d of n fulfils $1 < d < n$.

Result. 56, 76, 122

Solution. Denote by $s(n)$ the sum of all non-trivial divisors of a positive integer n. It is easy to analyse that there is no solution of the problem, if n has three or more different prime factors: $s(2 \cdot 3 \cdot 5) = 41$, $s(2 \cdot 3 \cdot 7) = 53$ and higher values of n give $s(n)$ greater than 63. Furthermore, if n were a power of a single prime number p, $s(n)$ would be divisible by p, so $p = 3$ or $p = 7$. However, we may easily check that no power of 3 or 7 satisfies the given condition.

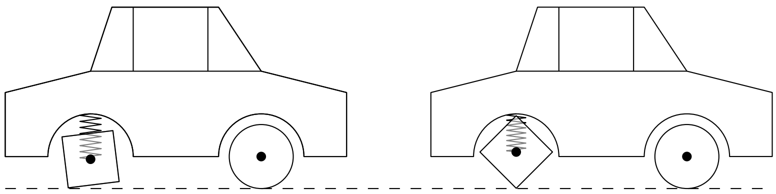
Therefore n possesses exactly two different prime factors p, q and a possible solution can be expressed as $n = p^\alpha \cdot q^\beta$. If $\alpha = \beta = 2$, then $p = 2$, $q = 3$ gives $s(n) = 54$ and all other combinations of prime factors or greater exponents result in $s(n) > 63$. Therefore at least one of the two exponents has to be smaller than 2; in particular, n is not a square of an integer, and as such, it has an even number of (non-trivial) divisors. Thus if all the divisors were odd, $s(n)$ would be an even number, which is not the case—we infer that (w.l.o.g.) $q = 2$.

For the same reasons, the number of odd non-trivial divisors (which are precisely $p, p^2, \ldots, p^\alpha$) is odd, hence α is odd. However, even for $p = 3$, $\alpha = 3$, and $\beta = 1$, $s(n)$ exceeds 63, therefore $\alpha = 1$. We conclude that

$$s(n) = 2 + 2^2 + \cdots + 2^\beta + p + 2p + \cdots + 2^{\beta-1}p = (2^\beta - 1)(p + 2).$$

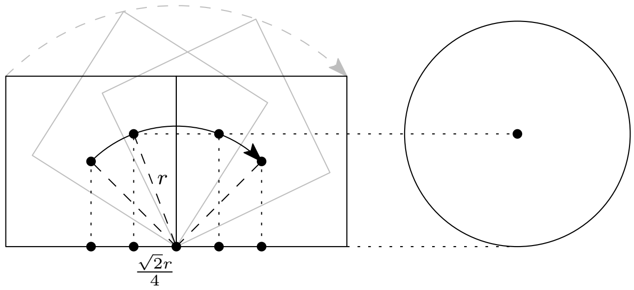
Out of all divisors of 63, only 1, 3, and 7 are of the form $2^\beta - 1$. These yield 61, 19, and 7 as corresponding values of p and so the possible values of n are $2 \cdot 61 = 122$, $2^2 \cdot 19 = 76$, and $2^3 \cdot 7 = 56$.

Problem 50. Bob has a cool car with square-shaped rear wheels (the front wheels are the standard round ones). Such a car would usually be rather unpleasant to drive, but Bob has installed very good shock absorbers for the rear wheels so that the car stays in a fixed position parallel to the surface while driving on a flat road. The side length of the rear wheel is 40 cm and its axle is fixed with respect to the car in the horizontal direction. What is the radius of the front wheel (in cm) provided that when the car is moving with constant forward speed, the axle of the rear wheel is exactly half of the time lower and half of the time higher from the surface than the axle of the front wheel?



Result. $10\sqrt{7}$

Solution. Consider the trajectory of the center of the square as the car moves forward—it consists of quarter-circles centered in a vertex of the square.



If the car maintains constant forward speed, this quarter-circle can also be viewed as the graph of the height of the rear axle as a function of time. Therefore the sought radius of the circular wheel is the height precisely in one quarter of the horizontal distance. Let r be the radius of the arc; then this one quarter equals $\sqrt{2}r/4$. Using the Pythagoras theorem, we obtain that the height at that moment is

$$\sqrt{r^2 - \left(\frac{\sqrt{2}}{4}r\right)^2} = \sqrt{\frac{7}{8}}r.$$

The result follows by plugging in $r = 20\sqrt{2}$.

Problem 51. The City of the Future has the shape of a regular 2017-gon. In vertices of the city there are 2017 metro stations, labelled $1, 2, \ldots, 2017$ counterclockwise on the city map. There are two metro lines: *side* and *diagonal*. The side line provides direct connection from station a to b (but not in the opposite direction) if and only if $a - b + 1$ is divisible by 2017 and one such ride lasts 1 minute. The diagonal line provides direct connection from station a to b if and only if $2b - 2a + 1$ is divisible by 2017 and one ride lasts 15 minutes. Ferdinand is an avid metro traveller and starting from station 1 he wants to get to the station n with the following property: the shortest possible time needed for getting to this station is the longest among all the stations. Find values n of all possible destinations of Ferdinand.

Result. 1984, 1985

Solution. Whenever Ferdinand travels one stop by diagonal line and then one stop by side line, he reaches the same destination in the same time as if he first took side and then diagonal line. Therefore we may w.l.o.g. assume that first he uses only diagonal line and then only side line. Further, if his journey consisted of at least two diagonal rides and at least one side ride, it could be replaced by a less time-consuming one (while keeping the destination) by omitting the last two diagonal and the first side ride, which together form an empty move in the metro network. Hence we may consider only journeys either using only diagonal line or using it at most once.

Note that if $n \geq 1009$, then using diagonal line once and side line $n - 1009$ times, Ferdinand will get to station n in $15 + (n - 1009)$ minutes. Obviously, using side line only cannot result in a shorter journey. On the other hand, if travelling only by diagonal line, he can reach station n in $2 \cdot (2018 - n) \cdot 15$ minutes. Thus we want to find $n \geq 1009$ for which $M(n) = \min\{n - 994, 30 \cdot 2018 - 30n\}$ is the largest possible. We have

$$n - 994 \leq 30 \cdot 2018 - 30n \iff n \leq \frac{30 \cdot 2018 + 994}{31} = \frac{31 \cdot 2018 - 1024}{31} = 2018 - 33 - \frac{1}{31} = 1985 - \frac{1}{31}.$$

Hence, for $n \leq 1984$: $M(n) = n - 994 \leq 1984 - 994 = 990$ and for $n \geq 1985$: $M(n) = 30 \cdot (2018 - n) \leq 30 \cdot (2018 - 1985) = 990$, so the desired largest minimum equals 990 and is achievable for $n = 1984$ and $n = 1985$.

It remains to check that for $n < 1009$ it is possible to get to station n in less than 990 minutes: for $n \leq 990$ Ferdinand can travel in $n - 1$ minutes only by side and for $n \geq 991$ he can travel in $15 \cdot (2 \cdot (1009 - n) + 1) \leq 15 \cdot 37 = 555$ minutes only by diagonal.

Problem 52. Let $f(n)$ be the number of positive integers that have exactly n digits and whose digits have a sum of 5. Determine, how many of the 2017 integers $f(1), f(2), \ldots, f(2017)$ have the units digit equal to 1.

Result. 202

Solution. Each n-digit number with digit sum of 5 can be represented as 5 ones assigned to some of n places of that number. Each place can have assigned more ones and the first place (from the left) has to have assigned at least 1 one. So the count of n-digit numbers with digit sum of 5 is equal to number of ways to distribute 4 ones to n places. In other words, we are computing number of 4-combinations of n elements with repetition, therefore

$$f(n) = \binom{n + 4 - 1}{4} = \frac{(n+3)(n+2)(n+1)n}{24}.$$

In the following, f will refer to this expression instead of the original combinatorial definition.

Let us now count the number of $f(n)$ whose unit digit is 1. To start with, if n has remainder after division by five 0, 2, 3, or 4, then n, $n+3$, $n+2$ or $n+1$ is divisible by 5. Since 24 and 5 are coprime, $f(n)$ is also divisible by 5, so its last digit is 0 or 5. Therefore the last digit of $f(n)$ can be 1 only when n has remainder 1 after division by 5.

Further, let us observe that $f(n)$ and $f(n+40)$ have the same units digit. Indeed, when computing $f(n+40)-f(n)$ and looking only at the numerators, we may proceed by expanding the expression

$$24f(n+40) = \big(40+(n)\big)\big(40+(n+1)\big)\big(40+(n+2)\big)\big(40+(n+3)\big),$$

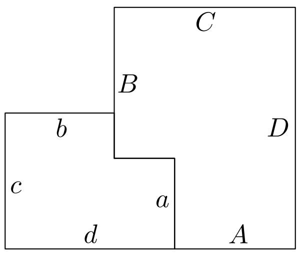
leaving the inner brackets unexpanded. After subtracting the term $24f(n) = n(n+1)(n+2)(n+3)$, the remaining terms are products of four numbers, out of which either at least two are 40, or exactly three are the brackets containing n. In the former case the term is divisible by 40^2, in the latter at least two of the brackets are consecutive integers, hence the term is divisible by 80. Therefore, after dividing by $24 = 8 \cdot 3$, the difference is divisible by 10 as desired. Consequently, it suffices to check the units digit of $f(n)$ only for any set of 40 consecutive integers n.

Finally, direct verification shows that

$$f(n) = f(-3-n). \tag{$\star$}$$

Taking into account all the facts, we need only to compute $f(1) = 1$, $f(6) = 126$, $f(11) = 1001$, and $f(16) = 3876$. Identity ($\star$) now implies that among the remaining four numbers of the form $f(5k+1)$, i.e. $f(-4)$, $f(-9)$, $f(-14)$, and $f(-19)$, two have units digit 1 and two have 6. Therefore $4 \cdot 2000/40 = 200$ of the numbers $f(1), \ldots, f(2000)$ have units digit of 1 and among $f(2001), \ldots, f(2017)$, this holds for $f(2001)$ and $f(2011)$. In total, there are 202 such numbers.

Problem 53. The picture shows two (indirectly) similar hexangular figures having some sides of length a, b, c, d and A, B, C, D, respectively. If these two figures are put together as in the picture, we get a new hexangular figure being similar to each of the two smaller ones (directly similar to the right-hand one). Find the ratio $A : a$.



Result. $\sqrt{\frac{1+\sqrt{5}}{2}}$

Solution. Let us call the three similar figures small, middle and large, respectively. Denote by p the sought ratio. Due to the similarity of the small and the middle figure we get

$$p = A : a = B : b = C : c = D : d.$$

Furthermore we easily see the equations $D = B + a$ and $C + b = A + d$. The ratio $B : A$ in the middle figure corresponds to the ratio $C : c$ in the large one. Hence we have $B : A = p$ and $B = p^2 a$ as a consequence. By analogous observations in the middle and the large figures we get $b : a = C : B = D : C$ and therefore

$$d : a = \frac{d}{c} \cdot \frac{c}{b} \cdot \frac{b}{a} = p^3.$$

Plugging in all these results into equation $D = B + a$ we get $pd = p^2 a + a = a(p^2+1)$ and thus $p^4 = p^2 + 1$. The only positive solution of the equation $p^4 - p^2 - 1 = 0$ as a quadratic equation with unknown p^2 is $p^2 = \frac{1+\sqrt{5}}{2}$. As a consequence we get

$$p = \sqrt{\frac{1+\sqrt{5}}{2}},$$

the square root of the golden ratio as the sought result.

Problem 54. Determine all pairs of positive integers (a, b) such that all the roots of both the equations

$$x^2 - ax + a + b - 3 = 0,$$
$$x^2 - bx + a + b - 3 = 0$$

are also positive integers.

Result. $(2, 2)$, $(6, 6)$, $(7, 8)$, $(8, 7)$

Solution. Let k, l be the roots of the first equation and m, n the roots of the second one. It is easy to see that if we have one solution (k,l,m,n), we can swap k and l or m and n or both and get another solution—therefore we will write only one of these solutions. According to Vieta's formulas,

$$k+l=a, \quad m+n=b, \quad kl=mn=a+b-3.$$

Combining these we obtain

$$kl+mn=2a+2b-6=2k+2l+2m+2n-6,$$

which can be rearranged to

$$(k-2)(l-2)+(m-2)(n-2)=2.$$

If both summands $(k-2)(l-2)$ and $(m-2)(n-2)$ are positive, i.e. equal to 1, we obtain solutions $(k,l,m,n)=(3,3,3,3)$ and $(1,1,1,1)$. If one of the summands is zero, there are solutions $(k,l,m,n)=(2,6,3,4)$ and $(3,4,2,6)$.

The remaining case is that one of the summands is negative; for this to happen, one of k, l, m, n has to equal 1. W.l.o.g. let $k=1$, then $l=mn$ and we can modify the equation to

$$2=-(l-2)+(m-2)(n-2)=-mn+2+mn-2m-2n+4=-2m-2n+6$$

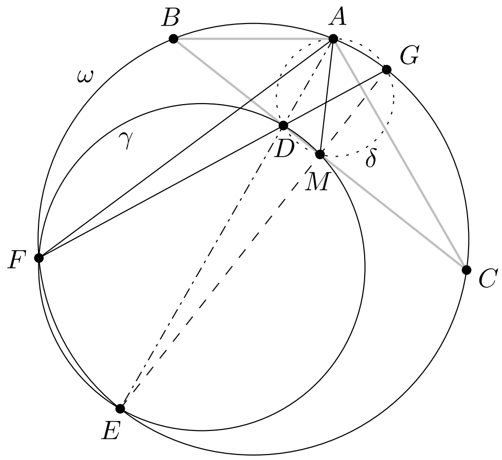
or $m+n=2$, which yields $m=n=1$ and $l=mn=1$. We infer that there in no solution with a negative summand.

Finally, the possible values of $(a,b)=(k+l,m+n)$ are $(6,6)$, $(8,7)$, $(7,8)$, $(2,2)$. We can easily check that all of these values of (a,b) satisfy the conditions from the statement.

Problem 55. Triangle ABC with $AB=3$, $BC=7$, and $AC=5$ is inscribed in circle ω. The bisector of angle $\angle BAC$ meets side BC at D and circle ω at a second point E. Let γ be the circle with diameter DE. Circles ω and γ meet at E and a second point F. Determine the length of AF.

Result. $\frac{30}{\sqrt{19}}$

Solution. It is a well-known fact that the perpendicular bisector of segment BC passes through point E. Let that bisector meet circle ω at second point G (so that EG is a diameter of ω) and M the midpoint of BC.



Using Thales' theorem in circles ω and γ, respectively, we infer that $\angle GFE=\angle DFE=90°$, implying that points G, D, F are collinear. Further, $\angle GMD=90°$ and $\angle GAD=\angle GAE=90°$ (Thales' theorem again) shows that the points D, M, G, A are concyclic; denote their common circle by δ. Now using inscribed angles (in the circles written above the equality sign) leads to

$$\angle AFD=\angle AFG\overset{\omega}{=}\angle AEG=\angle AEM$$

and

$$\angle FAD=\angle FAE\overset{\omega}{=}\angle FGE=\angle DGM\overset{\delta}{=}\angle DAM=\angle EAM.$$

It follows that triangles AFD and AEM are similar. In the same fashion we deduce

$$\angle ACD=\angle ACB\overset{\omega}{=}\angle AEB$$

which, together with $\angle DAC=\angle EAB$ (AE is angle bisector), proves the similarity of triangles CAD and EAB. Therefore

$$AF=AE\cdot\frac{AD}{AM}=AE\cdot\frac{AB\cdot\frac{AC}{AE}}{AM}=\frac{AB\cdot AC}{AM}.$$

The length of AM can be computed via Median formula as

$$AM = \frac{1}{2}\sqrt{2(AB^2 + AC^2) - BC^2} = \frac{1}{2}\sqrt{19}$$

and we conclude that

$$AF = \frac{3 \cdot 5}{\frac{1}{2}\sqrt{19}} = \frac{30}{\sqrt{19}}.$$

Problem 56. Determine the number of ordered triples (x, y, z), where x, y, z are non-negative integers smaller than 2017 such that

$$(x + y + z)^2 - 704xyz$$

is divisible by 2017.

Result. $2017^2 + 1 = 4\,068\,290$

Solution. Condition $2017 \mid (x + y + z)^2 - 704xyz$ can be rewritten as $(x + y + z)^2 \equiv 704xyz \pmod{2017}$. In the following, all congruences are considered modulo 2017. Since 2017 is a prime, for each positive integer $a < 2017$ there exist exactly one positive integer smaller than 2017, denoted here by a^{-1}, satisfying $a \cdot a^{-1} \equiv 1$.

First, let us find the triplets (x, y, z) where all three x, y, and z are non-zero. Let $y \equiv kx$ and $z \equiv lx$ (such k and l always exist: $k \equiv y \cdot x^{-1}$, $l \equiv z \cdot x^{-1}$). Substituting in the condition from the statement yields $(x + kx + lx)^2 \equiv 704klx^3$ and after multiplying by $(x^{-1})^2$ we get $(1 + k + l)^2 \equiv 704klx$. Finally, we multiply the congruence by $(704kl)^{-1}$, resulting in

$$x \equiv (704kl)^{-1}(1 + k + l)^2.$$

This implies that for each $k, l \in \{1, 2, \dots, 2016\}$ there exists exactly one x satisfying the condition from the statement (all adjustments were invertible). We can choose such k and l in 2016^2 ways. However, x has to be non-zero—this occurs if and only if $k + l \not\equiv 2016$. There are 2015 such pairs (k, l), one for each non-zero k excluding 2016. Thus there are $2016^2 - 2015$ desired triplets (x, y, z) with $x, y, z \neq 0$.

If $x = 0$ and y and z are non-zero, then we get $(y + z)^2 \equiv 0$ which holds if and only if $y \equiv -z$; there are 2016 such triplets $(0, y, z)$. In the same fashion we get 2016 triplets for $y = 0$ and $z = 0$. If two of the integers x, y, z equal zero, then the third one has to be zero, too, so we get one more triplet $(x, y, z) = (0, 0, 0)$.

To sum up, the total number of desired triplets (x, y, z) is

$$2016^2 - 2015 + 3 \cdot 2016 + 1 = 2016^2 + 2 \cdot 2016 + 1 + 1 = 2017^2 + 1.$$

Problem 57. Jane and Thomas play a game with a fair dice, which has two faces painted red, two green, and two blue. They alternately roll the dice until one of them has seen all three colors during his/her turns, this player being declared the winner. With what probability does Jane win the game, provided that she is the first one to roll the dice?

Result. $81/140$

Solution. Denote by $P_1(x, y)$ the probability that the player who is to roll the dice wins the whole game, provided that he/she has seen x colors and the other player y colors so far. Let $P_2(x, y)$ be the same probability for the currently non-rolling player (so that $P_1(x, y) + P_2(x, y) = 1$). Our goal is to compute $P_1(1, 1)$, i.e. the probability of the starting player to win right after both players took their first turn.

Since $P_2(2, 2) = \frac{2}{3}P_1(2, 2)$, it follows that $P_1(2, 2) = \frac{3}{5}$, $P_2(2, 2) = \frac{2}{5}$. Furthermore, from

$$P_2(2, 1) = \tfrac{2}{3}P_1(1, 2),$$
$$P_1(1, 2) = \tfrac{1}{3}P_2(2, 1) + \tfrac{2}{3}P_2(2, 2)$$

we obtain $P_1(1, 2) = \frac{12}{35}$, $P_2(1, 2) = \frac{23}{35}$, $P_1(2, 1) = \frac{27}{35}$, and $P_2(2, 1) = \frac{8}{35}$. Finally, we have

$$P_1(1, 1) = \tfrac{1}{3}P_2(1, 1) + \tfrac{2}{3}P_2(1, 2),$$

yielding $P_1(1, 1) = \frac{81}{140}$.

Problem 58. Given the equation

$$k(k+1)(k+3)(k+6) = n(n+1),$$

find the largest integer n for which there exists an integer solution (k, n).

Result. 104

Solution. Firstly, observe that given a solution (k, n), the only different solution with the same value of k is $(k, -1-n)$. Of the two integers n, $-1-n$, one is always non-negative and the other one negative. As we are interested in the largest value of n, we may further assume that $n \geq 0$. For these numbers the expression $n(n+1)$ is increasing as a function of n, hence to get n as large as possible, we have to maximize the left-hand side of the equality.

Denote by $P(k)$ the left-hand side polynomial $k(k+1)(k+3)(k+6) = k^4 + 10k^3 + 27k^2 + 18k$. We will use the fact that between two subsequent numbers $n(n+1)$ and $(n+1)(n+2)$ of the right-hand side there is no other number of this kind. Let us therefore try to approximate $P(k)$ by a polynomial $(k^2 + ak + b)(k^2 + ak + (b+1))$ with variable k and integer coefficients a and b. Comparing the coefficient of k^3, we start with $a = 5$. Expanding yields

$$\begin{aligned}(k^2 + ak + b)(k^2 + ak + (b+1)) &= (k^2 + 5k + b)(k^2 + 5k + (b+1)) \\ &= k^4 + 10k^3 + (26 + 2b)k^2 + (10b + 5)k + (b^2 + b).\end{aligned}$$

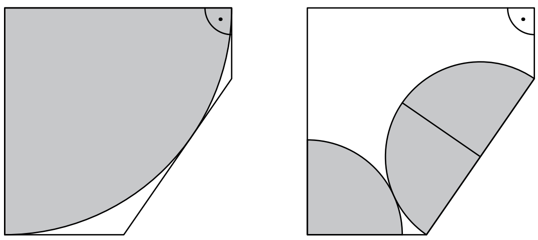
Now there is no appropriate integer value for b fulfilling $26 + 2b = 27$. Therefore, plugging in $b = 0$, $b = 1$ respectively, we get for sufficiently large absolute values of k the inequalities

$$k^4 + 10k^3 + 26k^2 + 5k \overset{(1)}{<} k^4 + 10k^3 + 27k^2 + 18k \overset{(2)}{<} k^4 + 10k^3 + 28k^2 + 15k + 2.$$

For numbers of this kind, $P(k)$ is strictly between two consecutive numbers of the form $n(n+1)$. Hence we have to analyze for which values of k the two inequalities are valid. Inequality (1) simplifies to $0 < k^2 + 13k = k(k+13)$, hence $k > 0$ or $k < -13$. Similarly, (2) gives $0 < k^2 - 3k + 2 = (k-1)(k-2)$, so $k > 2$ or $k < 1$. Thus for $k < -13$ or $k > 2$ both inequalities are fulfilled and $P(k)$ is between two consecutive numbers of the form $n(n+1)$. Therefore we have to analyze only the situations with $-13 \leq k \leq 2$.

Replacing the inequalities (1), (2) above by equalities, one gets satisfactory values of k, since then $P(k)$ equals a number of the form $n(n+1)$; therefore there exist solutions (k, n) for $k = -13, 0, 1, 2$. As $P(k)$ is increasing for $k \geq 0$ and we are searching for the maximal appropriate value of $P(k)$, $P(0)$ and $P(1)$ are not significant. Similarly, since $P(k)$ is clearly decreasing for $k \leq -6$, it remains to check $k = -5, -4, -3, -2, -1$. However, $P(k) \leq 0$ for $-6 \leq k \leq -3$ and $-1 \leq k \leq 0$, so only $P(-2)$ is of interest. It is easy to see that out of values $P(-13)$, $P(-2)$, $P(2)$, the first one is the greatest. For $k = -13$, (1) becomes equality, therefore $n = k^2 + 5k = 104$ is the sought number.

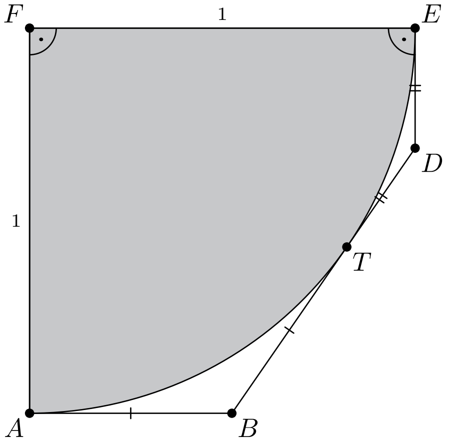
Problem 59. The Quarter-Pizzeria delivers pizza in special pentagonal boxes, which are suitable both for a quarter of a big pizza and for three quarters of a small pizza (as shown in the picture). What is the radius of a small pizza (in cm) if the radius of a big pizza is 30 cm?

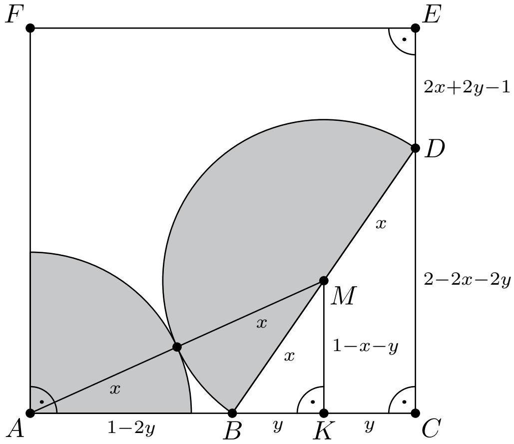


Result. $5(1 + \sqrt{7} - \sqrt{2\sqrt{7} - 4})$

Solution. Let $ABDEF$ be a pentagon similar to the described box with $EF = 1$ (taking 30 cm as a unit, we receive the problem's conditions). Note that $AF = EF$ (radii of a big pizza) and $\angle AFE = \angle BAF = 90°$ (central angles in circle quarters), so there exists a point C such that $ACEF$ is a square. Let K, M be the midpoints of segments BC,

BD, respectively and let T be the point of tangency of the big pizza quarter to the segment BD.





Let x be the sought radius of a small pizza, i.e. $BD = 2x$. Several tangencies give $AM = 2x$ and $AB = BT$, $DE = DT$, so $AB + DE = BD$. If $BC = 2y$, then $AB = 1 - 2y$, $DE = BD - AB = 2x + 2y - 1$, $CD = 2 - 2x - 2y$ and $KM = CD/2 = 1 - x - y$. Applying Pythagorean theorem to BKM and AKM, we get

$$y^2 + (1 - x - y)^2 = x^2 \quad \text{and} \quad (1 - y)^2 + (1 - x - y)^2 = 4x^2.$$

Thus,

$$y^2 + 4x^2 - (1 - y)^2 = x^2, \quad \text{so} \quad y = \frac{1 - 3x^2}{2}.$$

Plugging this into one of the above equations, we get after simplification

$$9x^4 - 6x^3 - 2x + 1 = 0.$$

Note that

$$\begin{aligned} 9x^4 - 6x^3 - 2x + 1 &= (3x^2)^2 + (x - 1)^2 - 2 \cdot 3x^2(x - 1) - 7x^2 \\ &= (3x^2 - x + 1)^2 - 7x^2 \\ &= (3x^2 + (\sqrt{7} - 1)x + 1)(3x^2 - (\sqrt{7} + 1)x + 1), \end{aligned}$$

so we obtain an equation

$$(3x^2 + (\sqrt{7} - 1)x + 1)(3x^2 - (\sqrt{7} + 1)x + 1) = 0.$$

It is now easy to see that the first factor has no roots over the reals and the second one has roots of the form

$$\frac{1}{6}\left(1 + \sqrt{7} \pm \sqrt{2\sqrt{7} - 4}\right).$$

However, the larger of the two roots is readily seen to be greater than $1/2$ which contradicts the inequality

$$2BD = BD + AB + DE < BC + CD + AB + DE = 2.$$

This means that $x = \frac{1}{6}(1 + \sqrt{7} - \sqrt{2\sqrt{7} - 4})$ and so the answer is

$$30x = 5\left(1 + \sqrt{7} - \sqrt{2\sqrt{7} - 4}\right) \text{cm}.$$