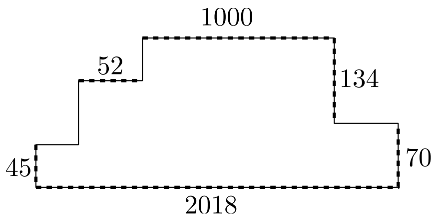
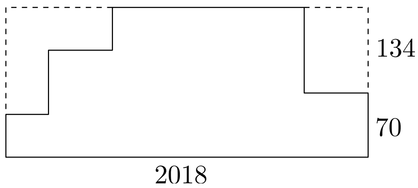
Problem 1. The picture shows a decagon with all sides meeting at right angle. Lengths of some of the sides (shown as dashed in the picture) are known and given in centimeters.



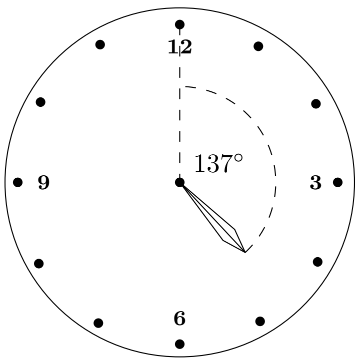
What is the perimeter of the decagon in cm?

Result. 4444

Solution. By swapping the "inner" corners into "outer" corners, the decagon can be transformed into a rectangle of dimensions 2018 and $70 + 134 = 204$. Therefore the perimeter is $2 \cdot (2018 + 204) = 4444$.



Problem 2. The minute hand of this clock is missing. How many minutes have passed since the last full hour, if the angle between the hour hand and twelve o'clock is 137°?



Result. 34

Solution. Since the hour hand passes $360° : 12 = 30°$ in one hour, it takes 2 minutes to pass 1°. Therefore, it passed $137° - 4 \cdot 30° = 17°$ after four o'clock, which took $17 \cdot 2 = 34$ minutes.

Problem 3. Four students, Kevin, Liam, Madison, and Natalie, took an exam. We know that their scores were 2, 12, 86, and 6 in some order. We also know that

- Kevin's score was *pampam* than Madison's score,
- Madison's score was *pampam* than Liam's score,
- Natalie's score was *pampam* than Liam's score,
- Kevin's score was *pampam* than Natalie's score.

where *pampam* means either "greater" or "smaller" (the same meaning in all four cases). What was the sum of Madison's and Natalie's scores?

Result. 18

Solution. One can see that if *pampam* stands for greater, then Kevin has the largest score and Liam has the smallest score, and if *pampam* means smaller, it is just the other way round. In any case, Madison and Natalie always have the middle two scores, namely 6 and 12. Therefore the sought sum is 18.

Problem 4. Jack and John are standing in a square and counting houses around. However, each starts counting (clockwise) at a different house, so Jack's house no. 4 is John's 16, and Jack's 12 is John's 7. How many houses are there in the square?

Result. 17

Solution. Since Jack's 4 is John's 16, there is a segment of houses where John's numbering is larger by 12. However, this segment has to end before Jack reaches 12, since otherwise John would get $12 + 12 = 24$. Since the number always drops by the total number of houses when the end is reached, we see that there are $24 - 7 = 17$ houses on the square.

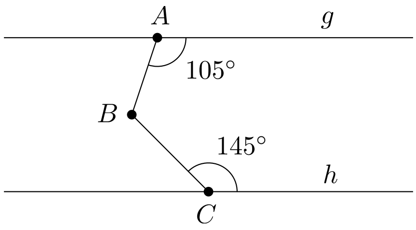
Problem 5. Doris has to decalcify the coffee machine. According to the instruction manual, she should mix four parts of water and one part of a 10% vinegar concentrate. Unfortunately, she can only find a bottle of 40% vinegar concentrate in her cupboard. How many parts of water have to be mixed with one part of 40% vinegar concentrate in order to get the prescribed concentration for decalcifying the coffee machine?

Note: Vinegar concentrate of n% consists of n parts of vinegar and $100 - n$ parts of water.

Result. 19

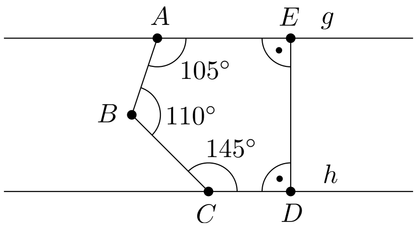
Solution. In the original recipe, the vinegar makes 10% of one of 5 parts. The same concentration is achieved if the vinegar makes $40\% = 4 \cdot 10\%$ of one of $4 \cdot 5 = 20$ parts. Hence we need 19 extra parts of water.

Problem 6. If g is parallel to h and the angles at A and C are $105°$ and $145°$ as indicated in the picture, what is the measure of angle $\angle CBA$?

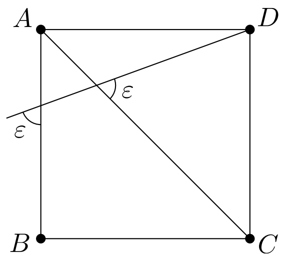


Result. 110°

Solution. We can add points D, E lying on h, g, respectively, so that the known angles and the angle in question become the internal angles of pentagon $ABCDE$. Since the sum of the newly added angles is $180°$ (we can make them right as in the figure below, but that is not necessary) and the sum of internal angles of any pentagon is $540°$, we infer that the sought value is $540° - 180° - 105° - 145° = 110°$.



Problem 7. If $ABCD$ is a square, what is the measure of angle ε (in degrees)?



Result. 67.5°

Solution. Let X, Y be the two points with angle ε. Then $\angle AXY = \angle AYX = \varepsilon$. Further, since $\angle XAY = \angle CAB = 45°$, the interior angles of triangle XYA satisfy

$$45° + \varepsilon + \varepsilon = 180°$$

or $\varepsilon = 67.5°$.

Problem 8. Cederic was born to his mother when she celebrated her 27th birthday. At most how many times can it happen that Cederic's age is the same as his mother's age read backwards?

Note: Possible leading zeros of numbers are ignored, e.g. 470 read backwards is 74.

Result. 7

Solution. Let Cederic be c years old and his mother m years old, c being equal to m reversed. The numbers c and m have the same number of digits (with c possibly having a leading zero, if m ends with a zero), which is at least 2. Let a and b be the units digit of c and m, respectively. As Cederic's mother is 27 years older, we see that either $a + 7 = b$ or $a + 7 = 10 + b$. If the mother was at least 100 years old, the difference of the first digits of their ages could be at most 1, which is not possible, as those are precisely the digits b and a. Thus, both numbers c and m have 2 digits.

Therefore, we want to find all numbers $\overline{ab}$ such that

$$\overline{ab} = \overline{ba} + 27.$$

We know that $a > b$, so the condition $a + 7 = b$ cannot hold. Hence we consider the condition $a + 7 = 10 + b$ or $a = b + 3$. Since $a \leq 9$, we have $b \leq 6$. For every digit $b \in \{0, 1, \ldots, 6\}$, we get digit a as $a = b + 3$. It is easy to see that for these digits the equality $\overline{(b+3)b} = \overline{b(b+3)} + 27$ holds. So the desired situation can happen 7 times: When Cederic has 3, 14, 25, 36, 47, 58, and 69 years.

Problem 9. Julia uses 32 white and 32 black cubes of side length 1 to form one big cube of dimensions $4 \times 4 \times 4$. She wants that the surface of the new cube contains as many white faces of the unit cubes as possible. What is the maximum possible proportion of the big cube's surface area that is white?

Result. 3/4

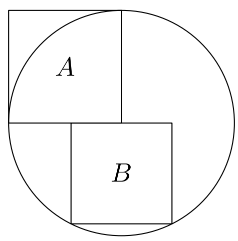
Solution. If a unit cube is placed in a corner of the big cube, then three of its faces are visible, if it is on one of the edges, two faces are visible, otherwise only one face is shown. There are eight corners and each of the twelve edges of the big cube consists of two unit cubes—altogether 32 places, and it is clear that the highest proportion of white area is achieved if the white cubes are placed precisely to these places. In this placement every face of the big cube looks the same, showing twelve white and four black faces. Hence the proportion for the whole surface is equal to $12/16 = 3/4$.

Problem 10. One hundred people took part in the selection of the crew for a flight to Mercury. Each potential astronaut underwent three tests checking certain health, psychological, and experience criteria. Only twenty-six people passed the health check successfully. Moreover, sixty participants failed more than one of the three tests. Finally, there were eighty-three people failing either the psychological or the experience test, but nobody failed these two simultaneously. How many of the participants were chosen for the mission, i.e. passed all the three tests?

Result. 3

Solution. Since nobody failed in both the psychological and the experience test, all the participants failing in at least two tests must have failed because of their health. This gives us $(100 - 26) - 60 = 14$ people failing only the health check. Together with 83 people failing in psychology or experience, there were 97 participants dismissed, hence only 3 astronauts from the whole group were selected.

Problem 11. Square A has two sides that coincide with the radii of a circle, square B has two vertices on the same circle and shares a part of its edge with square A. Find the ratio of the area of square A to the area of square B.

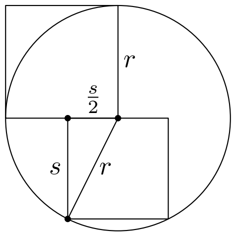


Result. $5 : 4$

Solution. Denote by s the side length of square B. Observe that by symmetry, the midpoint of the circle divides the side of square B which lies on the diameter in two equal parts of length $s/2$. Then the Pythagorean theorem yields

$$r^2 = \left(\frac{s}{2}\right)^2 + s^2 = \frac{5}{4}s^2$$

and therefore the ratio is $5 : 4$.



Problem 12. Determine the last two digits of the product

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37.$$

Result. 10

Solution. Since there is $2 \cdot 5$ in the product, the units digit will be 0. We obtain the tens digit as the last digit of the product $3 \cdot 7 \cdot 11 \cdot \ldots \cdot 37$. It is sufficient to consider only the units digits of multiplicands. Moreover, we can ignore ones. So we have to determine the last digit of the product

$$3 \cdot 7 \cdot 3 \cdot 7 \cdot 9 \cdot 3 \cdot 9 \cdot 7 = 3 \cdot 7 \cdot 3 \cdot 7 \cdot 3 \cdot 7 \cdot 9 \cdot 9.$$

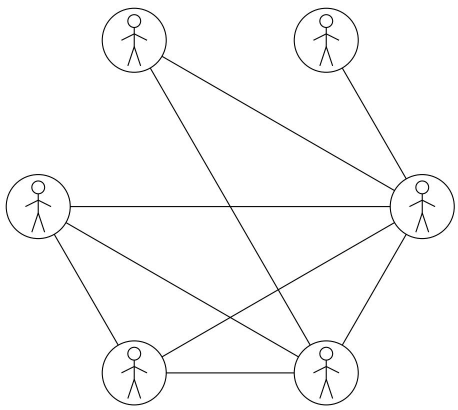
As $3 \cdot 7 = 21$ has the units digit 1, we can remove pairs of 3 and 7. After this, there is only $9 \cdot 9$ left, whose units digit is again 1. Therefore, the last two digits of the given product are 10.

Problem 13. A detective interrogated first five of six suspects of a crime. He found out that they have 1, 2, 3, 4, and 5 friends among all the six suspects, respectively. He knows that friendship is symmetric and decided to figure out the number of friends of the last suspect before his interrogation. How many was it?

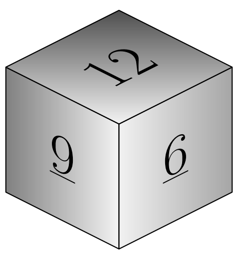
Result. 3

Solution. Let n be the number of friends of the last suspect. The suspect with five friends is a friend of everyone else, so omitting him from the group simply decreases everyone's number of friends by one. Then we can also omit the suspect with one friend only, since his number of friends dropped to zero and he is no longer significant in any way. This way we obtain a group of four suspects, about which we know that they have 1, 2, 3, and $n-1$ friends among themselves, respectively. Repeating the two steps produces yet smaller group with values 1 and $n-2$; it is clear then that $n-2=1$ or $n=3$.

Note: This solution can be turned to a way of producing such a group of suspects. The result is shown in the following diagram:



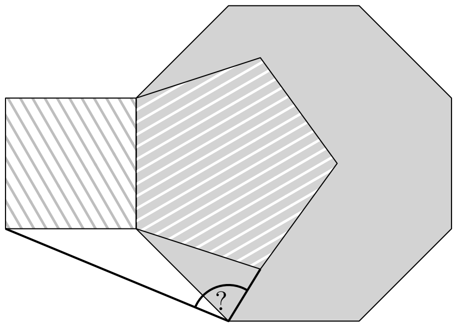
Problem 14. The die in the picture has a positive integer written on each of its faces. Moreover, the product of the numbers on opposite faces is the same for all such pairs. The numbers on the faces do not have to be distinct. What is the smallest possible value of the sum of all the numbers on the die?



Result. 40

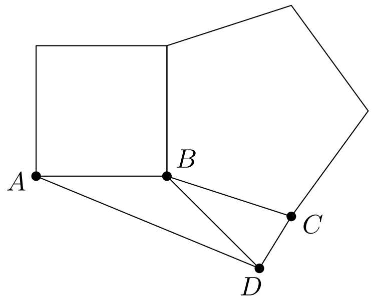
Solution. Let P be the product of numbers on opposite faces. Clearly, the higher P, the higher the total sum. Since P has to be divisible by all the three shown numbers, its smallest value of P is their least common multiplier, $P = 36$. Under these circumstances, the numbers not shown are 3, 4, and 6 and the sought sum is $6 + 9 + 12 + 6 + 4 + 3 = 40$.

Problem 15. If the grey octagon and the striped pentagon are regular, and the striped quadrilateral is a square, determine the measure of the angle between the thick segments.



Result. $99°$

Solution. Let us denote the points as in the picture.



Angle CBD is the difference of interior angles of octagon and pentagon, hence $\angle CBD = 135° - 108° = 27°$. We also easily obtain that $\angle ABD = 135°$. Since both triangles ABD and CBD are isosceles,

$$\angle CDB = \tfrac{1}{2}(180° - \angle CBD) = 76.5°,$$
$$\angle BDA = \tfrac{1}{2}(180° - \angle ABD) = 22.5°.$$

Therefore

$$\angle CDA = \angle CDB + \angle BDA = 99°.$$

Problem 16. The minister has a personal driver who leaves the ministry at fixed time in the morning to pick up the minister at his place and take him to the ministry. The minister wakes up at the same time every day and the car comes exactly when he is ready to go. Today, the minister woke up early and he was ready to leave one hour earlier than usual, so he decided to walk towards the car (which departed from the ministry as usual). He met the car, got in and arrived at the ministry twenty minutes earlier than usual. How many minutes did he spend walking? Assume that the car moves always at the same speed and that it takes no time for the minister to get in the car.

Result. 50

Solution. The time the minister gained by getting up earlier (1 hour) splits into the unknown length t of the walk and the time which would remain for the car to get from the meeting point to the minister's house, which is obviously half of the total saved time, therefore

$$60 = t + \frac{20}{2}$$

or $t = 50$.

Problem 17. What is the smallest positive integer, which has at least two digits and when its first (i.e. leftmost) digit is erased, the value drops 29 times?

Result. 725

Solution. Let d be the first digit of the number, k the number obtained after erasing the first digit, and n the number of digits of k. Then the original number equals $10^n d + k$ and the assertion can be rewritten as

$$10^n d + k = 29k$$

or

$$28k = 10^n d.$$

Since $28 = 2^2 \cdot 7$, in order for the right-hand side to be divisible by 28, $d = 7$ and $n \geq 2$ has to hold. Finally, the choice $n = 2$ (implying $k = 25$) gives the smallest possible number 725.

Problem 18. How often in 24 hours is the minute hand of a clock perpendicular to its hour hand?

Result. 44

Solution. The minute hand does 24 revolutions in 24 hours, the hour hand does 2 revolutions in 24 hours. Therefore the minute hand laps the hour hand 22 times in 24 hours. In each of this 22 times the minute hand and the hour hand are perpendicular 2 times, therefore the answer is 44.

Problem 19. Find all four-digit palindrome numbers that can be written as a sum of two three-digit palindromes.

Note: A palindrome is a number which stays the same when the order of its digits is reversed, e.g. 2018102 is a palindrome. A number cannot begin with a zero.

Result. 1111, 1221

Solution. Let $\overline{abba}$ be such a palindrome. Since it is a sum of two three-digit numbers, it does not exceed 1998, so $a = 1$. Let $\overline{1bb1}$ be equal to $\overline{cdc} + \overline{xyx}$, then

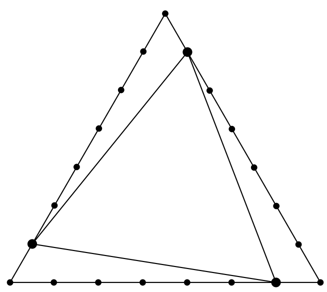
$$1001 + 110b = 101(c + x) + 10(d + y).$$

Since the left-hand side ends with 1, $c+x$ also ends with one. As both c and x are at least one and at most 9, $c+x = 11$. Plugging in and simplifying produces

$$11(b - 1) = d + y.$$

Since d and y are digits, the right-hand side does not exceed 18, so $b - 1$ is either 0 or 1. Both options are possible: $1111 = 505 + 606$, $1221 = 565 + 656$.

Problem 20. The sides of an equilateral triangle are divided into two segments that are in the ratio of 6 to 1 in such a way that the dividing points also form an equilateral triangle (see figure). Determine the ratio of the area of the smaller equilateral triangle to the area of the larger equilateral triangle.



Result. 31/49

Solution. The area of each of the three small triangles is

$$\frac{1}{7} \cdot \frac{6}{7} = \frac{6}{49}$$

of the large equilateral triangle, because the height is 1/7 and the base 6/7 of the corresponding lines in the large equilateral triangle. Therefore the ratio of the area of the smaller equilateral triangle to the area of the larger equilateral triangle is

$$1 - 3 \cdot \frac{6}{49} = \frac{31}{49}.$$

Problem 21. Find all quadruples (a, b, c, d) of positive integers such that when we replace the letters in the table below by the assigned values, then a, b, c, d will be exactly the number of ones, twos, threes, and fours present in the table, respectively.

1	2	3	4
a	b	c	d

Result. $(2, 3, 2, 1)$, $(3, 1, 3, 1)$

Solution. No number can appear in the table more than five times; however, the number five cannot appear, since it would take a position used by the number appearing five times. Hence only the numbers 1, 2, 3, and 4 can be filled in.

Let us show that $d = 1$: If $d = 2$, then one of a, b, c has to be 4, and because there are only two free slots, it has to be b. However, $(2, 4, 2, 2)$ is clearly not a valid quadruple. The options $d = 3$ and $d = 4$ lead even faster to a contradiction.

We now know that $a \in \{2, 3\}$. Assuming that $a = 2$, we get $b, c \in \{2, 3\}$ (there cannot be any other ones and fours), but $b = 2$ would contradict the condition on b (there would be three twos), hence $b = 3$, and $c = 2$ follows directly. If $a = 3$, there has to be one more one in the table, and that cannot be c (there is already another three), so $b = 1$, and again $c = 3$ is immediate.

Problem 22. Peter forgot his password. He only remembers that the password consisted of nine lower-case latin letters and contained words 'math' and 'drama'. How many passwords do satisfy these requirements?

Note: The words are contained as substrings, therefore, for example, 'martha' does not contain 'math'. There are altogether 26 letters in the alphabet.

Result. 2030

Solution. Firstly, consider the case when the subwords 'math' and 'drama' do not overlap. Then there are two ways to order them: 'dramamath' and 'mathdrama'.

If they do overlap, there is only one possible way to arrange them: 'dramath'. There are three ways to choose places for the two remaining letters: '∗∗dramath', '∗dramath∗', 'dramath∗∗'. In every case, there are $26^2 = 676$ ways to choose these two letters. Hence, in these cases, there are $676 \cdot 3 = 2028$ possible passwords.

In total, there are $2028 + 2 = 2030$ possible passwords.

Problem 23. If one chooses two arbitrary distinct numbers from the set $\{1, 2, 3, \ldots, n-1, n\}$, the probability that the numbers are consecutive positive integers is $\frac{1}{21}$. Determine n.

Result. 42

Solution. There are $n - 1$ pairs of consecutive numbers in the set $\{1, 2, 3, \ldots, n-1, n\}$ and there are $\frac{1}{2}n(n-1)$ possibilities to choose two arbitrary different numbers. Therefore we have

$$\frac{n-1}{\frac{1}{2}n(n-1)} = \frac{2}{n} = \frac{1}{21}$$

yielding $n = 42$.

Problem 24. Arthur, Ben and Charlie were playing table tennis using the following rules: Each round, two players played against each other and the remaining one rested. The winner of the round then played in the next round with the rested player. In the first round, Arthur played against Ben. After several rounds, Arthur had scored 17 victories and Ben 22. How many times did Arthur and Ben play against each other?

Result. 20

Solution. Observe that whenever Charlie wins a round, it has no impact on the number of rounds won by Arthur or Ben, neither has it any impact on the number of rounds when Charlie does not play. Hence we may assume that Charlie always loses. In other words, every victory of Arthur over Ben increases Arthur's overall score by two (unless it happened in the last round), and vice versa. Since the number of Arthur's victories is odd, we see that the last round had to be Arthur vs. Ben, won by Arthur. Therefore, if we add one more round (Arthur vs. Charlie won by Arthur), the total number of rounds when Charlie did not play would be one half of the sum of final scores of Arthur and Ben, i.e. $(18 + 22)/2 = 20$.

Problem 25. E-shop customers can express their satisfaction with a purchased item by rating it online using a five-point rating scale (1 star = poor, 5 stars = excellent). The average rating of a newly released smartphone was 3.46 stars last week, however, as two more people submitted their ratings earlier this week, it rose to the current average of 3.5 stars. How many people have rated the smartphone so far?

Result. 52

Solution. Denote k the original number of ratings and x their sum. Further denote a, b the two this week's ratings. Then

$$\frac{x}{k} = 3.46 \qquad \text{and} \qquad \frac{x+a+b}{k+2} = 3.5$$

or

$$x = \left(3 + \tfrac{23}{50}\right) k, \tag{1}$$

$$x + a + b = \left(3 + \tfrac{1}{2}\right) k + 7. \tag{2}$$

Equation (1) implies that k is a multiple of 50. Moreover, after subtracting (1) from (2), we get

$$a + b - 7 = \frac{k}{25}.$$

As $a, b \leq 5$, the left-hand side is a positive integer not exceeding 3, hence $k \leq 75$. We conclude that $k = 50$ and after adding the two customers rating this week, we see that 52 people have rated so far.

Problem 26. Juliette has four pairs of socks with Monday, Tuesday, Wednesday, Thursday written on each single sock. How many ways are there to wear all of these socks from Monday to Thursday, if the two socks on Juliette's feet should be different and neither of them showing the current day? None of the socks can be worn repeatedly.

Note: Any sock can be worn on any foot, i.e. there are no "right" and "left" socks. Furthermore, wearing a sock on the right foot and another sock on the left foot counts the same as wearing them reversed.

Result. 9

Solution. For the sake of brevity, we will use numbers 1, 2, 3, 4 instead of the names of the days. Observe that each day is assigned three distinct numbers: The actual number of the day and the two numbers of the socks being worn. Therefore, we may equivalently describe the assignment of socks by a single number for each day—the only number out of the four not appearing in the aforementioned triple. We infer that the valid assignments of socks correspond to the rearrangements of $(1, 2, 3, 4)$, i.e. permutations which leave none of the numbers at its original position.

The number of rearrangements can be computed as follows: There are three options of placing 1, let $n \neq 1$ be its position. Now n has also three options where to be put. It is easy to see that the remaining two numbers are now assigned in an unique way, hence there are $3 \cdot 3 = 9$ rearrangements and the same number of choices of Juliette's socks.

Problem 27. A jury of 26 mathematicians was to nominate (at least) five films for awards at a festival of math-themed films. There were 16 films to choose from. The jury had chosen the following procedure: Each jury member voted for five distinct films and the five films with most votes were nominated; if there was a tie on the fifth place, all these films were nominated. What is the smallest number of votes that a film could have received so that it was nominated no matter the results of other films?

Result. 21

Solution. In total, $26 \cdot 5 = 130$ votes were distributed among the films. On one hand, if a film received 20 or less votes, the remaining 110 votes can easily be distributed so that there are five films getting 21 votes each. On the other hand, if the film received at least 21 votes, it not being nominated would imply at least five films getting at least 22 votes, resulting in at least $21 + 5 \cdot 22 = 131$ votes cast, a contradiction.

Problem 28. A real function f satisfies $f(x) + xf(1-x) = x$ for every real value of x. Find $f(-2)$.

Result. $4/7$

Solution. From the equation $f(-2) - 2f(3) = -2$ we see that it is equivalent to determine $f(3)$ instead. Since $f(3) + 3f(-2) = 3$, we have two linear equations for the unknown values $f(-2)$ and $f(3)$. Multiplying the second equation by 2 and adding both equations we get $f(-2) = 4/7$.

Problem 29. Two-digit numbers n, a, b, o, j are such that their product $naboj$ is divisible by 4420. Determine the greatest possible value of their sum $n + a + b + o + j$.

Result. 471

Solution. Firstly, let us factorize $4420 = 2 \cdot 2 \cdot 5 \cdot 13 \cdot 17$. Since 13 and 17 are primes, one of the numbers n, a, b, o, j has to be divisible by 13 and one by 17. As the smallest common multiple of 13 and 17 is 221, there is no two-digit number divisible by both of them. Without loss on generality, we can assume that n is divisible by 17 and a is divisible by 13. This means that $n \leq 85 = 5 \cdot 17$ and $a \leq 91 = 7 \cdot 13$.

Suppose now that $n = 85$ and $a = 91$. We see that $n = 85$ is divisible by 5, so we only need to guarantee the divisibility by 4. As n and a are odd, 4 has to divide boj. Therefore, one of the numbers b, o, j is divisible by 4, or there are two numbers divisible by 2. We get the greater sum in the second case, when $b = o = 98$ and $j = 99$. We have found the sum $n + a + b + o + j = 85 + 91 + 98 + 98 + 99 = 471$.

Finally, let us check the possibility that $n < 85$ or $a < 91$. Since the numbers n and a have to be divisible by 17 and 13, respectively, this would mean $n \leq 68 = 85 - 17$ or $a \leq 78 = 91 - 13$. Then the sum $n + a + b + o + j$ could be at most $68 + 91 + 3 \cdot 99 = 456$ (in the former case) or $85 + 78 + 3 \cdot 99 = 460$ (in the latter case) and that is less than we have previously achieved.

Problem 30. Naomi ordered eight tennis balls and one handball at an online sports shop. The balls (of a perfect spherical shape) were packed in a cubic box so that each tennis ball was tangent to three of the six faces of the box and to the handball. The radius of a handball is 10 cm and the radius of each tennis ball is 5 cm. Find the length of the edge of the box in centimeters.

Result. $10(1 + \sqrt{3})$

Solution. The space diagonal of the box passes through the centers of the handball and two tennis balls, and also through the points of tangency of these three balls. The only area where the diagonal is not inside any ball are the segments between a tennis ball and corner of the box; the distance from the center of a tennis ball to the corner is one half of the space diagonal of the circumscribed cube of the ball. So the length of the diagonal of the box is the sum of

- (2×) 1/2 of the diagonal of a cube circumscribed to a tennis ball,
- (2×) the radius of a tennis ball,
- the diameter of the handball.

Hence the length of the diagonal is equal to

$$10\sqrt{3} + 10 + 20 = 30 + 10\sqrt{3},$$

and the length of the edge is equal to

$$\frac{30 + 10\sqrt{3}}{\sqrt{3}} = 10(1 + \sqrt{3}).$$

Problem 31. Written in the decimal system, the power 2^{29} is a nine-digit number whose digits are pairwise distinct. Which digit is missing?

Result. 4

Solution. On one hand, the power 2^{29} can be computed by hand with reasonable effort: For example, use $2^{10} = 1024$, compute 1024^2 and $1024^2 \cdot 1024$. Finally, divide the result by 2 to get $2^{29} = 536\,870\,912$.

On the other hand, you can use the fact that an integer and its digital sum have the same residue class modulo 9. Moreover, the residue class of 2^n modulo 9 is periodic with period of length 6. Since the sum of all digits is 45, we end up having

$$45 - x \equiv 2^{29} \equiv 2^5 \equiv 5 \pmod 9$$

where x denotes the missing digit in the decimal representation of 2^{29}. This leads to $x \equiv 4 \pmod 9$. Therefore the missing digit is 4.

Problem 32. When cleaning the attic, Ben found an old calculator, which showed only the first two digits after the decimal point for each result, but was able to compute square roots. So for example, for $\sqrt{4}$ the machine displayed 2.00 and for $\sqrt{6} = 2.44949\ldots$ it showed 2.44. What is the smallest positive integer, which is not a square of an integer, but for whose square root would Ben's calculator show two zeros after the decimal point?

Result. 2501

Solution. Denote by $\operatorname{ftd}(n)$ the first two digits after the decimal point of $\sqrt{n}$. It is clear that as n increases from one square number to the next one, $\operatorname{ftd}(n)$ increases as well; since we are looking for the smallest integer n, this has to be of the form k^2+1 for some positive integer k.

When $\sqrt{k^2+1}$ is rounded down to its integer part, the result is k, therefore $\sqrt{k^2+1}-k$ is a number strictly between 0 and 1. The assertion that $\operatorname{ftd}(k^2+1) = 0$ can thus be equivalently restated as

$$\sqrt{k^2+1}-k < \frac{1}{100}.$$

Adding k to both sides, squaring (both sides are positive) and rearranging yields

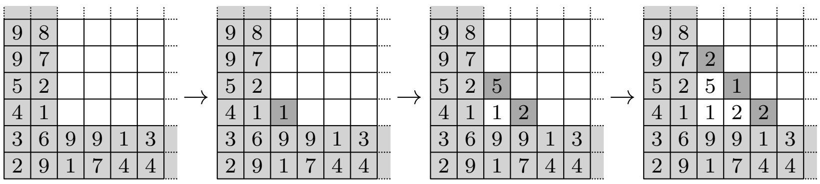
$$k > 50\left(1 - \frac{1}{100^2}\right).$$

The right-hand side is a number strictly between 49 and 50; since k is an integer, $k \geq 50$ follows. We conclude that $n = 50^2 + 1 = 2501$ is the sought smallest number.

Problem 33. One of the numbers from 1 to 9 is to be written in each cell of a 2018×2018 board in such a way that within any 3×3 square, the sum of the inputs is divisible by 9. In how many different ways can this be accomplished?

Result. 9^{8068}

Solution. Any filling of the 8068 cells forming the two bottom rows and the two leftmost columns defines the whole filling explicitly as we can uniquely fill in the remaining numbers on consecutive diagonals—see the example in the picture. On the other hand, obviously, each correct filling induces some filling of the 8068 cells. Therefore the number of arbitrary fillings of those cells is the same as the sought number of correct fillings of the table.



Problem 34. Find all pairs of positive integers (n, m) fulfilling the equation $4^n + 260 = m^2$.

Result. $(3, 18)$, $(6, 66)$

Solution. The given equation is equivalent to $m^2 - (2^n)^2 = 260$ and factorizing the left-hand side leads to $(m - 2^n)(m + 2^n) = 260$. The prime factorization of $260 = 2^2 \cdot 5 \cdot 13$ leads to the following possible decompositions:

$$260 = 1 \cdot 260 = 2 \cdot 130 = 4 \cdot 65 = 5 \cdot 52 = 10 \cdot 26 = 13 \cdot 20,$$

taking into account that $(m - 2^n) < (m + 2^n)$. Due to $(m + 2^n) - (m - 2^n) = 2^{n+1}$ we get the two possibilities $26 - 10 = 2^4$ and $130 - 2 = 2^7$ leading to the two pairs $(3, 18)$ and $(6, 66)$ fulfilling the the given equation.

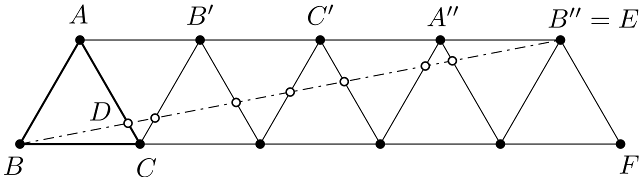
Problem 35. In an equilateral triangle ABC, a light ray coming from B hit AC at point D satisfying $DC : AC = 1 : 2018$, and it reflected such that the angle of incidence was equal to the angle of reflection. Then it reflected again every time it reached a side of $\triangle ABC$. How many times did it reflect (including the first reflection) until it reached some vertex of $\triangle ABC$?

Result. 4033

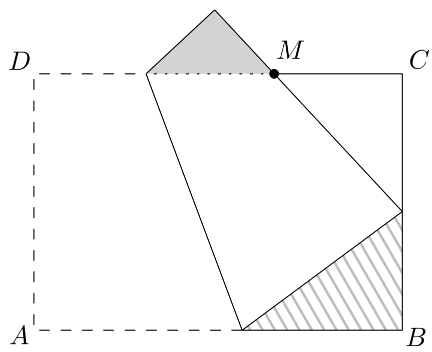
Solution. Instead of reflecting the light ray, we shall let it proceed straight and subsequently reflect the triangle along the side with which the ray is incident. Let us show that one row of these reflected triangles is sufficient for the light ray to reach a vertex of one of the triangles.

Let E be the intersection of the ray BD with line through A parallel to BC, and F a point on BC such that $EF \parallel AC$. Then the triangles BCD and BFE are similar and $BF = 2018BC$. This implies that the point E may be obtained by reflecting the triangle ABC and it is clearly first such point on BD.

It is easy to see that the segment BE intersects $2 \cdot 2017 - 1 = 4033$ segments in the row of triangles, which equals the number of reflections of the light ray in the original problem. The picture illustrates the solution with the initial condition changed to $DC : AC = 1 : 5$.

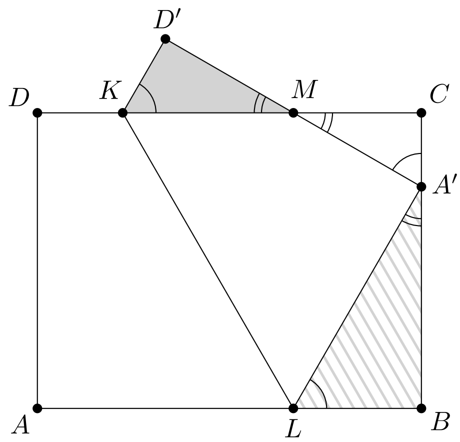


Problem 36. A rectangular sheet of paper $ABCD$ was folded so that (former) point A ended on side BC and point M, where side CD met (former) side DA, was exactly in one third of CD, i.e. $CD = 3CM$. If the area of the gray overlapping triangle is 1, what is the area of the striped triangle?



Result. 9/4

Solution. Let us denote the various intersection points as in the figure.



All the three triangles KMD', $A'MC$, and $LA'B$ are right and similar to each other, as straightforward angle chasing shows, hence we seek the ratio of similitude between the two triangles in question. Taking into account that $KD = KD'$ and $LA = LA'$, we have

$$D'K + KM = DM = \tfrac{2}{3}DC = \tfrac{2}{3}AB$$

and

$$A'L + LB = AB.$$

Since $A'L$ corresponds to KM and LB corresponds to KD' in the aforementioned similarity, this shows that the ratio is $3/2$. As we are interested in areas, the result is $(3/2)^2 = 9/4$.

Problem 37. When dividing the polynomial $x^3 + x^5 + x^7 + x^9 + x^{11} + x^{2017} + x^{2018}$ by the polynomial $x^2 - 1$ there is a remainder. Find the value of the variable x for which the numerical value of that remainder is 1111.

Result. 185

Solution. The polynomial can be rewritten as

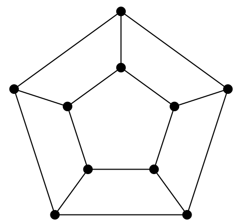
$$x^3 + x^5 + x^7 + x^9 + x^{11} + x^{2017} + x^{2018} =$$
$$= x(x^2 - 1) + x(x^4 - 1) + x(x^6 - 1) + x(x^8 - 1) + x(x^{10} - 1) + x(x^{2016} - 1) + (x^{2018} - 1) + 6x + 1.$$

Now, since

$$x^{2k} - 1 = (x^2 - 1)(x^{2k-2} + x^{2k-4} + \cdots + 1),$$

we see that all the brackets on the right-hand side of the equality are divisible by $x^2 - 1$. As the degree of $6x + 1$ is smaller than the degree of $x^2 - 1$, we infer that $6x + 1$ is the sought remainder. Now solving $1111 = 6x + 1$ produces the answer $x = 185$.

Problem 38. There are ten towns in the country of Pentagonia, each one connected by three railway lines to another three towns according to the diagram below. The antitrust laws of the country require that no two lines having a common stop are operated by the same railway company. In how many ways can the lines be assigned to three railway companies in a lawful way?

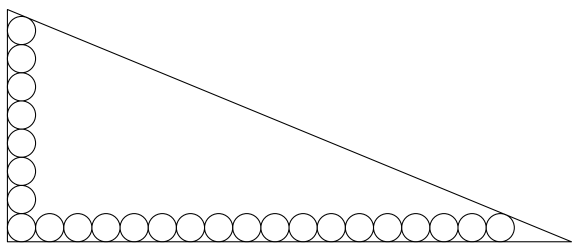


Result. 30

Solution. Observe that when assigning the lines on the "outer pentagon", two of the companies (denoted by X and Y) have to get two lines and one company (Z) obtains one line. The companies also have to be in order $XYXYZ$ starting from some town. It is easy to show that when we have assigned these lines, the rest of the network is assigned in a unique way: The lines of the "inner pentagon" are assigned to the same companies as their outer counterparts and the "connecting" lines must each go to the sole unused company.

This means that the number of correct assignments of the whole network is equal to the number of assignments of the outer pentagon. There are six ways to assign the three companies to X, Y, and Z, and five possibilities for the town where the assignment $XYXYZ$ starts from. This gives us $5 \cdot 6 = 30$ ways in total.

Problem 39. A right triangle contains 25 tangent circles of radius 1 as shown in the picture.



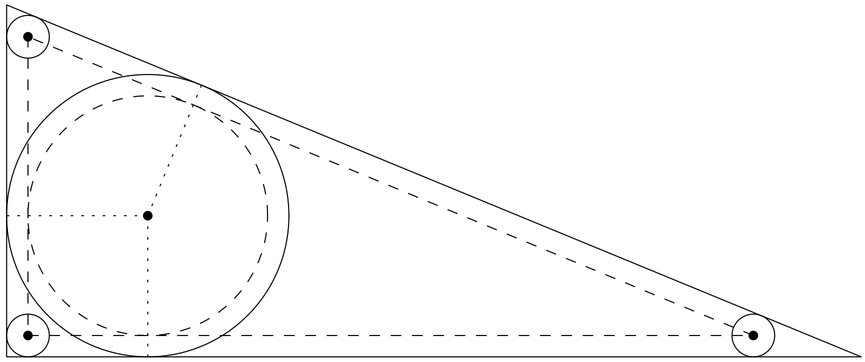
What is the radius of the incircle of the triangle?

Result. $25 - 13\sqrt{2}$

Solution. Consider the triangle whose vertices are centers of the three circles in the corner. This is a right triangle with legs of length 14 and 34, respectively; hence, by the Pythagorean theorem, the length of the hypotenuse is

$$\sqrt{14^2 + 34^2} = 26\sqrt{2}.$$

Further, its inradius can be computed as $(14 + 34 - 26\sqrt{2})/2 = 24 - 13\sqrt{2}$. Since the sides of the original triangle are parallel to the new ones and their distance is 1, the incenters of the two triangles coincide and the sought inradius is just the one we have computed plus the distance, i.e. $25 - 13\sqrt{2}$.



Problem 40. Marge has invented the operation *marging* on a (finite) list of integers: given a list, she takes four copies of it, increases their entries by 0, 2, 3, and 5, respectively, and concatenates the results, forming again a single list. For example, given the list $(8, 3)$, the result after marging is $(8, 3, 10, 5, 11, 6, 13, 8)$. If Marge starts with the one-element list (0) and keeps marging it until it has at least 2018 entries, what will be the 2018th entry? (The leftmost entry is considered to be the first one.)

Result. 17

Solution. For convenience, let us number the positions in the list starting with zero instead. In that case, an easy induction argument shows that the position of a number written in the base 4 system describes what operations (and in what order) were applied to obtain the number on the given position. For example, this is what Marge's list looks like after two iterations of marging:

$$(\underset{00}{0+0}, \underset{01}{0+2}, \underset{02}{0+3}, \underset{03}{0+5}, \underset{10}{2+0}, \underset{11}{2+2}, \underset{12}{2+3}, \underset{13}{2+5}, \underset{20}{3+0}, \underset{21}{3+2}, \underset{22}{3+3}, \underset{23}{3+5}, \underset{30}{5+0}, \underset{31}{5+2}, \underset{32}{5+3}, \underset{33}{5+5}).$$

(The numbers under the entries are their positions in base 4.) Since 2017 written in base 4 is 133201, the number on position 2017 is

$$2 + 5 + 5 + 3 + 0 + 2 = 17.$$

Problem 41. Determine the smallest positive integer n such that the equation

$$(x^2 + y^2)^2 + 2nx(x^2 + y^2) = n^2y^2$$

has a solution (x, y) in positive integers.

Result. 25

Solution. The equation can be also viewed as a quadratic equation in n with solution

$$n = \frac{(x^2 + y^2)\big(x + \sqrt{x^2 + y^2}\big)}{y^2}$$

(the other solution would lead to negative n since $\sqrt{x^2 + y^2} > x$), or

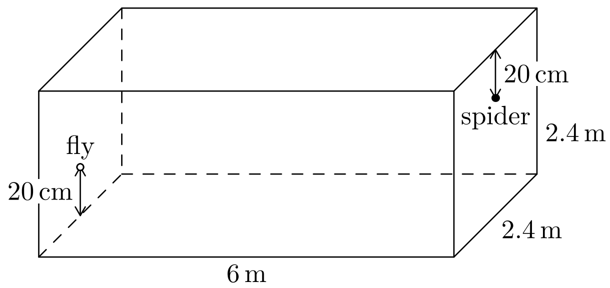
$$ny^2 = (x^2 + y^2)\big(x + \sqrt{x^2 + y^2}\big).$$

Let $d = \mathrm{GCD}(x, y)$ and let $x = x_0d$, $y = y_0d$. Plugging in and simplifying yields

$$ny_0^2 = d(x_0^2 + y_0^2)\Big(x_0 + \sqrt{x_0^2 + y_0^2}\Big).$$

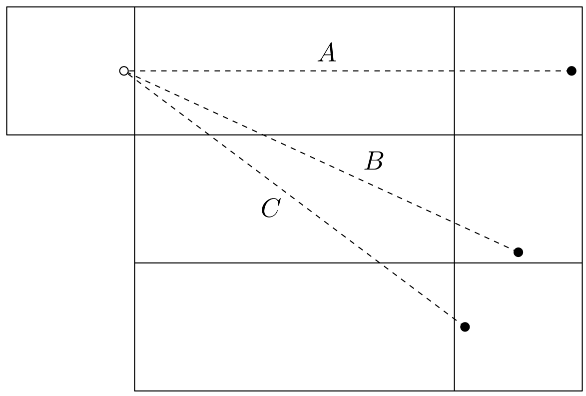
Now since x_0 and y_0 are coprime, also y_0^2 and $x_0^2 + y_0^2$ are coprime and $x_0^2 + y_0^2 \mid n$ follows. Further, the presence of $\sqrt{x_0^2 + y_0^2}$ forces $x_0^2 + y_0^2$ to be a square. It is well known that $5^2 = 25$ is the smallest square which is the sum of two squares, $3^2 + 4^2$. Hence $n \geq 25$ and plugging in $x = 4$, $y = 3$ shows that $n = 25$ indeed gives a solution.

Problem 42. In a rectangular room with dimensions 6 m × 2.4 m × 2.4 m (length × width × height), a spider is located on one 2.4 m × 2.4 m wall 20 cm away from the ceiling and with equal distance to the vertical edges. A fly, sitting on the opposite wall, is on its vertical axis of symmetry, too, but 20 cm away from the floor. If the fly does not move at all, what is the shortest total distance (in meters) the spider must crawl along the surfaces in order to capture the fly?



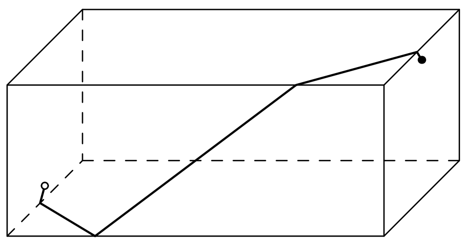
Result. 8

Solution. Let us examine the spider's possible paths in the net of the cuboid. It is clear that the shortest path becomes a straight line segment when the net is put into a plane. The white circle represents the fly, whereas the black circle stands for the spider, its location in the plane dependent on the way we decompose the net of the cuboid.



There are (up to a symmetry) three possible ways for the spider to reach the fly, crossing one, two, or three of the long faces of the cuboid; the paths are marked by A, B, and C in the picture. Clearly, a path using four of these faces could be reduced to a shorter one. The length of path A is 8.4 m and using the Pythagorean theorem, we obtain that the length of path B in meters is $\sqrt{66.32}$ and for path C it is 8. Therefore path C is the shortest one and the answer is 8 m.

The image below shows the shortest path in three dimensions:



Problem 43. Find the minimum of

$$(6 + 2\cos(x) - \cos(y))^2 + (8 + 2\sin(x) - \sin(y))^2$$

for $x, y \in \mathbb{R}$.

Result. 49

Solution. Let us set $V(x, y) = (6 + 2\cos(x) - \cos(y))^2 + (8 + 2\sin(x) - \sin(y))^2$. Recall that the circle with center (C_1, C_2) and radius $R > 0$ can be parametrized (i.e. coordinates of all points lying on it can be expressed) by angle α as $(x_1, x_2) = (C_1 + R\cos(\alpha), C_2 + R\sin(\alpha))$. Let us consider circles k_1, k_2 with centers $(0, 0)$, $(6, 8)$ and radii 1, 2, respectively. Then it follows from the Pythagorean theorem that $V(x, y) = AB^2$ where $A \in k_1$ with angle x and $B \in k_2$ with angle y. It follows that the minimum of $V(x, y)$ is the square of the distance of the closest pair of points on k_1 and k_2 and we can compute it using the distance of the centers and the radii of k_1 and k_2: $\sqrt{6^2 + 8^2} - 1 - 2 = 7$. Thus the minimum of $V(x, y)$ equals $7^2 = 49$.

Problem 44. What is the smallest positive integer such that its last (i.e. units) digit is 2, and if we move the last digit in front of the first digit, we get the double of the original number?

Result. 105 263 157 894 736 842

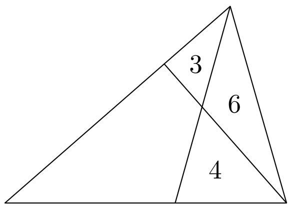
Solution. Let us call N the number in question. When we remove the units digit, we should get all the digits of $2N$ except for the leftmost one. Therefore, since N ends with 2, $2N$ has to end with 4, hence the tens digit of N is 4. Let d_i be the i-th digit of N, this time counting from right to left (i.e. d_1 is the units digit). Taking into account how multiplying by two works digit-wise, we see that the digits of N have to satisfy

$$d_i = \begin{cases} 2d_{i-1} \bmod 10 & \text{if } d_{i-2} < 5, \\ 2d_{i-1} \bmod 10 + 1 & \text{if } d_{i-2} \geq 5 \end{cases}$$

for all $i > 2$. This way we can directly write down the digits of N. We stop when we obtain the digit 1 and in the next step the digit 2: The number starting with this 1 is N, because multiplying by 2 precisely removes the last digit, but puts 2 in front of the number. The result is

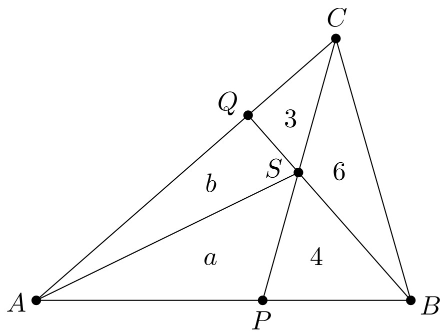
$$N = 105\,263\,157\,894\,736\,842.$$

Problem 45. Mother Berta divides her triangular shaped area of land with two straight lines into four pieces and gives the piece of size 6 to her daughter Betty, the one of size 4 to her daughter Barbara and the smallest one with size 3 to the youngest daughter Francis. She keeps the largest piece for herself. How big is this piece of land?



Result. 19/2

Solution. We use the notation as in the following picture.



Due to the area ratios, S divides QB in the ratio $1:2$ and PC in the ratio $2:3$. Introducing the straight line AS and denoting the area of triangle ASQ by b and the area of triangle APS by a leads to the following two equations:

$$\frac{b}{a+4} = \frac{1}{2}$$
$$\frac{b+3}{a} = \frac{3}{2}$$

These are equivalent to

$$2b = a + 4$$
$$2b + 6 = 3a,$$

which gives the solutions $a = 5$ and $b = \frac{9}{2}$. Therefore the area of quadrangle $APSQ$ is $\frac{19}{2}$.

Problem 46. Four brothers have altogether 2018 euros. It is known that the wealth of each of them is a positive integer, no two possess the same amount of euros, and whenever one brother is richer than another one, the wealth of the richer one is a multiple of the wealth of the poorer one. What is the smallest number of euros the richest brother could have had?

Result. 1152

Solution. Since each brother's wealth is a multiple of the wealth of the poorest one, their sum, 2018, has to be divisible by this number as well. However, prime factorization $2018 = 2 \cdot 1009$ gives only three options for the poorest one: 1, 2, or 1009. Clearly, 1009 is impossible, as this would be larger than any of the remaining three amounts. Further, if it was just 1, the rest would be left with 2017 euros, which is a prime number, hence the second poorest brother would have had only 1 as well—a contradiction. Therefore the poorest one has 2 euros and the remaining three have 2016 altogether.

Let $a < b < c$ be the fortunes of the three brothers; these are subject to the conditions $a \mid b \mid c$ and $a + b + c = 2016$. The divisibility together with strict inequality implies that $2a \leq b$ and $2b \leq c$; if we could achieve equalities, we would clearly get a solution with the smallest value of c. Luckily, $1 + 2 + 4 = 7$ divides 2016, therefore we can indeed divide this sum as

$$2016 = \tfrac{1}{7} \cdot 2016 + \tfrac{2}{7} \cdot 2016 + \tfrac{4}{7} \cdot 2016$$

and the answer is $\frac{4}{7} \cdot 2016 = 1152$.

Problem 47. Andrew drew the symbol ♣ on the blackboard. Then he repeated the following procedure thirteen times: He erased the blackboard and wrote a new sequence of symbols, having the pair ♣♡ instead of each ♡ and ♡♣ instead of each ♣ in the original sequence. For example, Andrew would replace the sequence ♣♡♡ with ♡♣♣♡♣♡. How many pairs ♡♡ (with no other symbol in between) were there on the blackboard when Andrew was finished with his task? The counted pairs may overlap, so e.g. in the sequence ♡♡♡♡, there are three ♡♡ pairs.

Result. 1365

Solution. Let A_n be the sequence on the blackboard after Andrew carried out the replacement procedure for the n-th time (with $A_0 = (\clubsuit)$) and h_n the number of pairs ♡♡ in A_n. Since each pair ♡♡ in A_n arises only from the pair ♡♣ in A_{n-1}, which, on the other hand, comes either from ♡♡ or ♣ in A_{n-2}, we see that $h_n = h_{n-2} + 2^{n-3}$ for $n \geq 3$, since there are exactly 2^{n-3} symbols ♣ in A_{n-2}. Therefore, for odd n, we have

$$h_n = 2^{n-3} + 2^{n-5} + \cdots + 2^0 + h_1 = \tfrac{1}{3}(2^{n-1} - 1),$$

since $h_1 = 0$. The desired result is $h_{13} = 1365$.

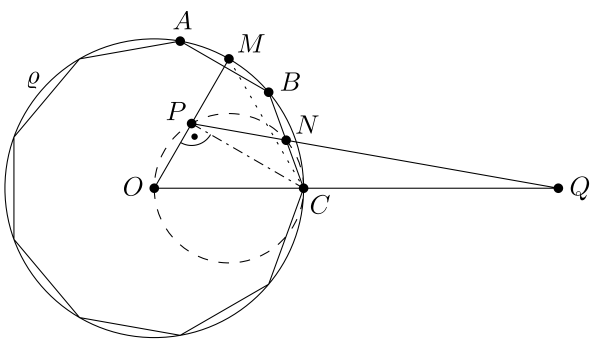
Problem 48. Let $ABCDEFGHI$ be a regular nonagon with circumcircle ϱ and center O. Let M be the midpoint of the (shorter) arc AB of ϱ, P the midpoint of MO, and N the midpoint of BC. Let lines OC and PN intersect at Q. What is the measure of $\angle NQC$ (in degrees)?

Result. $10°$

Solution. We will prove that the quadrilateral $OCNP$ is cyclic; since $\angle ONC = 90°$, this is equivalent to $\angle OPC = 90°$. This can be seen as follows: As both C and M lie on ϱ, $OC = OM$. Easy computation also shows that $\angle MOC = 60°$, so $\triangle OCM$ is equilateral. Now P, being the midpoint of OM, satisfies $\angle OPC = 90°$.

Another easy calculation reveals that $\angle OCN = 70°$, hence $\angle OPN = 180° - \angle OCN = 110°$. Using the triangle OQP, we get that

$$\angle NQC = \angle PQO = 180° - \angle POQ - \angle QPO = 10°.$$



Problem 49. Anna picked a triple (x, y, z) of positive integers such that $x + y + z = 2018$ and told x to Xena, y to Yena, and z to Zena. None of the three knew the other two numbers, but they were told the information about their sum. The following conversation followed:

- Xena: I know that Yena and Zena have different numbers.
- Yena: Thanks to Xena, now I know that all the three of us have different numbers!
- Zena: Now I can finally tell who was told what number.

Find the triple (x, y, z).

Result. $(3, 2, 2013)$

Solution. Xena's statement means just that x is odd; if it was even, y and z could have been the same.

Assume now that y is odd; that means that Yena knew from the beginning that x and z are different. If, moreover, $y \geq 1009$, Yena would have already known that x and z were different from y and thus would not need Xena's statement. On the other hand, if $y \leq 1007$, then despite Xena's statement Yena still could not tell if her number was different from x. We infer that y is even and consequently, z is odd.

If y was a multiple of 4, then $x + z = 2018 - y \equiv 2 \pmod 4$, i.e. it could be a double of an odd number; in such a case Yena could not deduce that x and z are different. However, if $y \equiv 2 \pmod 4$, then x and z have to give distinct remainders modulo 4 and Yena's statement is justified.

Finally, let us examine Zena's statement. Firstly, $y = 2$, for otherwise y could be decreased by 4 and x increased by 4 and Zena could not tell the difference. For similar reasons, $x \leq 4$, hence either $x = 1$ or $x = 3$. However, in the former case, knowing $2018 - z = x + y = 3$ Zena could have determined x and y without Yena's statement (using just what Xena had reported). We conclude that $x = 3$ and $z = 2013$.

Problem 50. Wizards Arithmetix and Combinatorica are engaged in a duel. Both wizards have 100 hit points (HP). Arithmetix's spell hits Combinatorica with probability 90% and deals 60 HP damage (if it succeeds), Combinatorica's spell hits Arithmetix with probability 60% and deals 130 HP damage. The wizards alternate in spell-casting, Arithmetix starts. The duel ends when a participant runs out of his or her hit points, the remaining wizard being the winner. Determine the probability that Arithmetix wins the duel.

Result. $45/128$

Solution. The exact amount of HP is not important—it clearly suffices to know that Arithmetix loses after being hit once and Combinatorica after being hit twice. Suppose that we are in the state of the duel when Combinatorica has been already hit once and it is Arithmetix's turn to cast. Denote the probability that Arithmetix wins by q. In this state, Arithmetix can win either if his attack succeeds, which happens with probability 0.9, or if he misses and so does Combinatorica in her turn—that happens with probability $0.1 \cdot 0.4$, and subsequently, Arithmetix wins again with probability q. Therefore, we obtain the equation

$$q = 0.9 + 0.1 \cdot 0.4 \cdot q,$$

which yields $q = 15/16$.

Let us now compute the probability p that Arithmetix wins the duel. If Arithmetix hits and Combinatorica misses (probability $0.9 \cdot 0.4$), the duel gets to the situation of the previous paragraph and Arithmetix wins with probability $q = 15/16$. On the other hand, if Arithmetix misses and Combinatorica misses, too (probability $0.1 \cdot 0.4$), then Arithmetix can again win with probability p. So we can write the equation

$$p = 0.9 \cdot 0.4 \cdot \frac{15}{16} + 0.1 \cdot 0.4 \cdot p.$$

Solving it shows that Arithmetix wins the battle with probability $p = 45/128$.

Problem 51. Let $a(1), a(2), \ldots, a(n), \ldots$ be an increasing sequence of positive integers satisfying $a(a(n)) = 3n$ for every positive integer n. Compute $a(2018)$.

Note: A sequence is *increasing* if $a(m) < a(n)$ whenever $m < n$.

Result. 3867

Solution. If $a(1) = 1$ we also have $a(a(1)) = 1 \neq 3 \cdot 1$ which is impossible. Since the sequence is increasing it follows that $1 < a(1) < a(a(1)) = 3$ and thus $a(1) = 2$. From the equation we deduce $a(3n) = a(a(a(n))) = 3a(n)$ for all n. We easily prove by induction (starting with $a(1) = 2$) that $a(3^m) = 2 \cdot 3^m$ for every m. Using this we also obtain $a(2 \cdot 3^m) = a(a(3^m)) = 3^{m+1}$.

There are $3^n - 1$ integers i such that $3^n < i < 2 \cdot 3^n$ and there are $3^n - 1$ integers j such that $a(3^n) = 2 \cdot 3^n < j < 3^{n+1} = a(2 \cdot 3^n)$. Since $a(n)$ is increasing there is no other option than $a(3^n + b) = 2 \cdot 3^n + b$ for all $0 < b < 3^n$. Therefore $a(2 \cdot 3^n + b) = a(a(3^n + b)) = 3^{n+1} + 3b$ for all $0 < b < 3^n$. Since $2018 = 2 \cdot 3^6 + 560$ we have $a(2018) = 3^7 + 3 \cdot 560 = 3867$.

Problem 52. Equilateral triangle T of side length 2018 is divided into 2018^2 small equilateral triangles of side length 1. We call a set M of vertices of these small triangles *independent* if for any two distinct points $A, B \in M$ the segment AB is not parallel to any side of T. What is the largest number of elements of an independent set?

Result. 1346

Solution. Each vertex in the grid can be assigned its distances to the three sides of T (taking as the unit the height of a small triangle); it is easy to see that for each vertex, these three integers add up to 2018. On the other hand, given a triple of non-negative integers with sum 2018, there is a unique vertex of the grid with these numbers being its distances from the sides, thus we may equivalently consider such triples instead of the vertices. We will refer to the three numbers as *coordinates*.

The independence condition translates to the assertion that no two triples in the set have equal the first, the second, or the third coordinate. Let

$$M = \{(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_k, y_k, z_k)\}$$

be an independent set. Since the numbers $x_1, \ldots, x_k$ are distinct non-negative integers, their sum is at least

$$0 + 1 + \cdots + (k-1) = \frac{k(k-1)}{2}.$$

The same holds for the sums $y_1 + \cdots + y_k$ and $z_1 + \cdots + z_k$. On the other hand, we have $x_i + y_i + z_i = 2018$ for each $i = 1, \ldots, k$, and so

$$3 \cdot \frac{k(k-1)}{2} \leq (x_1 + \cdots + x_k) + (y_1 + \cdots + y_k) + (z_1 + \cdots + z_k) = 2018k.$$

It follows that

$$k \leq 1 + \frac{2}{3} \cdot 2018$$

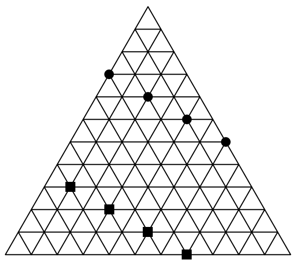
or $k \leq 1346$.

The following two sequences of points form together an independent set of size 1346:

$$(0, 672, 1346),\ (2, 671, 1345),\ (4, 670, 1344),\ \ldots,\ (1344, 0, 674);$$
$$(1, 1345, 672),\ (3, 1344, 671),\ (5, 1343, 670),\ \ldots,\ (1345, 673, 0).$$

We conclude that the sought maximal number of elements is 1346.

The following picture illustrates the construction of an independent set for a triangle of side length 11 instead of 2018:



Problem 53. Let ABC be a triangle with $AB = 5$, $AC = 6$ and ω its circumcircle. Let F, G be points on AC such that $AF = 1$, $FG = 3$, and $GC = 2$, and let BF and BG intersect ω in D and E, respectively. Given that AC and DE are parallel, what is the length of BC?

Result. $5\sqrt{5/2}$

Solution. Denote $x = BC$. Since $ACED$ is an isosceles trapezoid, we may put $y = AE = CD$. Finally, let $p = BF$, $q = DF$, $u = BG$, and $v = GE$.

Angles BAC and BDC are inscribed in the same circle and hence of the same size. Consequently, triangles ABF and DCF are similar, which implies

$$\frac{y}{5} = \frac{q}{1} = \frac{5}{p}.$$

Furthermore, in the same way we obtain the similarity between triangles BCG and AEG, from which

$$\frac{y}{x} = \frac{v}{2} = \frac{4}{u}$$

follows. Finally, as AC and DE are parallel,

$$\frac{p}{q} = \frac{u}{v}$$

and combining with the preceding equations yields

$$\frac{\frac{25}{y}}{\frac{y}{5}} = \frac{\frac{4x}{y}}{\frac{2y}{x}}$$

or $x^2 = 125/2$. Therefore, $x = 5\sqrt{5/2}$.

Problem 54. We know that

$$2^{22000} = \underbrace{4569878\ldots229376}_{6623\text{ digits}}.$$

For how many positive integers $n < 22000$ is it also true that the first digit of 2^n is 4?

Result. 2132

Solution. If the first digit of a k-digit number N is c, then $c10^{k-1} \le N < (c+1)10^{k-1}$. This implies that $2c10^{k-1} \le 2N < (2c+2)10^{k-1}$, i.e. the first digit of $2N$ is at least the first digit of $2c$ and at most the first digit of $2c+1$. We apply this to the first digits of powers of two: Having a power of two with the first digit equal to 1, there are these five possibilities for the first digits of the following powers of two: (1) $1, 2, 4, 8, 1$; (2) $1, 2, 4, 9, 1$; (3) $1, 2, 5, 1$; (4) $1, 3, 6, 1$; (5) $1, 3, 7, 1$.

Let k be a non-negative integer such that 2^k begins with 1 and has d digits. Then there is a unique power of two beginning with 1 and having $d+1$ digits, and it is either 2^{k+3} (if we are in one of the situations (3), (4), (5) above) or 2^{k+4} (given that the case (1) or (2) occurs). As 2^0 (having 1 digit) and 2^{21998} (having 6623 digits) begin with 1, we can compute how many times does (1) or (2) occur when computing successive powers of two: It is exactly $21998 - 3 \cdot 6622 = 2132$ times.

Finally, observe that the cases (1) and (2) are precisely those giving rise to a power of two starting with 4, therefore there are exactly 2132 such numbers in the given range.

Problem 55. Find rational numbers a, b, c such that

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

Note: A rational number is a quotient of two integers.

Result. $(1/9, -2/9, 4/9)$

Solution. Let us set $x = \sqrt[3]{\sqrt[3]{2} - 1}$ and $y = \sqrt[3]{2}$. The idea is to use the fact that the numbers $y^3 \pm 1$ are integers and employ the factorization identities for $A^3 \pm B^3$ to obtain a relation between x and y in a form suitable to express x as sum of three rational cube roots. Firstly note that

$$1 = y^3 - 1 = (y-1)(y^2 + y + 1)$$

and since $3 = y^3 + 1$ we have

$$y^2 + y + 1 = \frac{3y^2 + 3y + 3}{3} = \frac{y^3 + 3y^2 + 3y + 1}{3} = \frac{(y+1)^3}{3}.$$

Hence

$$x^3 = y - 1 = \frac{1}{y^2 + y + 1} = \frac{3}{(y+1)^3}.$$

Secondly, since

$$3 = y^3 + 1 = (y+1)(y^2 - y + 1)$$

we have

$$\frac{1}{y+1} = \frac{y^2 - y + 1}{3}$$

and finally

$$x = \frac{\sqrt[3]{3}}{y+1} = \sqrt[3]{\frac{1}{9}}(\sqrt[3]{4} - \sqrt[3]{2} + 1).$$

We have proved that the triple $(a, b, c) = (\frac{4}{9}, -\frac{2}{9}, \frac{1}{9})$ works.

It is possible to prove that this representation of x as sum of three cube roots of rationals is unique up to a reordering.