Problem 1. In a game for five players, one of them gets a point in every round. The game ends as soon as one of the players collects 10 points. How many rounds at most can the game have?

Result. 46

Solution. When the game ends, the winner has 10 points, while all the other players have at most 9 points, which sums up to $10 + 4 \cdot 9 = 46$.

Problem 2. Gleb has four cards with the numbers 1, 2, 3 and 6 written on them. He wants to arrange all the cards so that they form two numbers A and B such that A is a multiple of B, e.g. $A = 36$ and $B = 12$. In how many ways can he do it?

Result. 21

Solution. Let us consider two cases:

I. B contains one card and A contains three. Let us consider the possible values of B:

- $B = 1$: A is any permutation of $2, 3, 6 \to 6$ options.
- $B = 2$: A must end with $6 \to 2$ options.
- $B = 3$: A is any permutation, since the sum of digits is divisible by $3 \to 6$ options.
- $B = 6$: A must end with $2 \to 2$ options.

II. A and B both contain two cards. Then the ratio A/B will be less than 6, so it can be 1, 2, 3, 4, or 5. Let us work out these possibilities:

1: This is not possible, as it would imply $A = B$.

2: A must end with 2 or 6. If A ends with 2, then B must end with 6 or 1. In the former case, we get 32 and 16, in the latter, 62 and $31 \to 2$ options. If A ends with 6, then B must end with 3, we get 26 and $13 \to 1$ option.

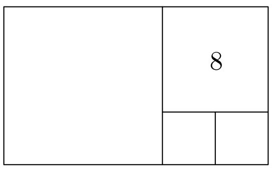
3: A must start with 6 or 3. In the former case, B must start with 2, we get 63 and 21. In the latter case, B must start with 1, we get 36 and $12 \to 2$ options.

4: A must start with 6 and end with 2, since it is even. So $A = 62$, and it is not divisible by 4.

5: A must end with 5 or 0, but this is not possible.

Summing everything up, we get 21 numbers.

Problem 3. The picture contains four squares, one of them has an area of 8. What is the area of the largest one?



Result. 18

Solution. The squares have side lengths in the ratio of $3 : 2 : 1$. Hence, the area of the largest square is $\left(\frac{3}{2}\right)^2 \cdot 8 = 18$.

Problem 4. There were once several magpies, mathematicians, and centaurs in a park. In particular, there were 15 tails and 94 hands in total. How many legs were there?

Note: Magpies have no hands, two legs and one tail, mathematicians have two hands, two legs, but no tail while centaurs have two hands, four legs and one tail.

Result. 124

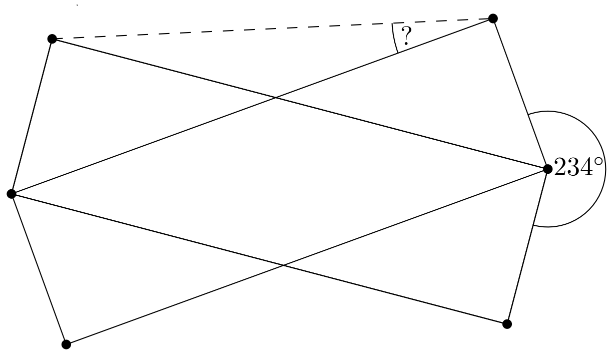
Solution. We denote the number of magpies, mathematicians, and centaurs by g, m and c. Then the condition on the number of tails implies $g + c = 15$, while the condition on the number of hands translates to $2m + 2c = 94$. The number of legs is $2g + 2m + 4c$, which is $2(g + c) + (2m + 2c) = 30 + 94 = 124$.

Problem 5. There are three channels on your TV: 1st, 2nd, and 3rd. From each channel, you can switch only to a channel with the number differing by one, e.g., from the 1st only to the 2nd. You start watching the second channel and then switch the channel 11 times. How many different sequences of channels can you obtain?

Result. 64

Solution. There will be twelve channels in the sequence. The channels in odd positions must be the 2nd for sure, and each of the channels in even positions could be either the 1st or 3rd. There are six such positions, so we get $2^6 = 64$.

Problem 6. There are two congruent rectangles and one given angle size in the picture. Determine the size of the angle labelled by the question mark in degrees.



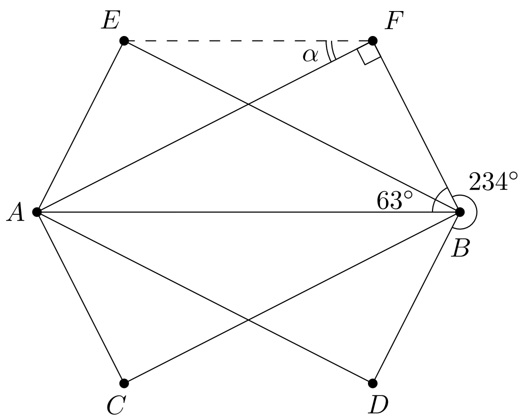
Result. 27°

Solution. Let us label the points as on the picture below. The common diagonal AB bisects the angle $\angle FBD$, hence

$$\angle FBA = \frac{360^\circ - 234^\circ}{2} = 63^\circ.$$

Moreover, since the dashed line EF is parallel to AB, we get for the sum of angles $\angle EFB + \angle FBA = 180^\circ$. Subtracting the known right angle $\angle AFB$, we get

$$\alpha = 180^\circ - 90^\circ - 63^\circ = 27^\circ.$$



Problem 7. What is the value of $x^3 - 14x + 2024$ if $x^2 - 4x + 2 = 0$?

Result. 2016

Solution. We take the value of interest $x^3 - 14x + 2024$ and subtract $x(x^2 - 4x + 2) = 0$ in order to cancel x^3, obtaining $4x^2 - 16x + 2024$. Now we want to cancel $4x^2$, so we subtract $4(x^2 - 4x + 2) = 0$ and obtain 2016 which is the answer.

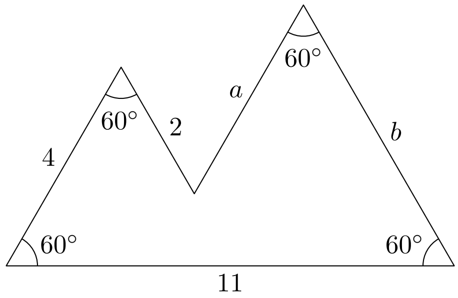
Problem 8. Michal chose a positive integer n and wrote down the number of its even digits, odd digits, and all digits in this order. Reading these three numbers as one positive integer from left to right and ignoring possible zeros on the left, he obtained n again. What is the smallest such n?

For example, if Michal took the number 2024, then the number of even digits is 4, the number of odd digits is 0, and the number of all digits is also 4, so he reads the outcome as 404.

Result. 123

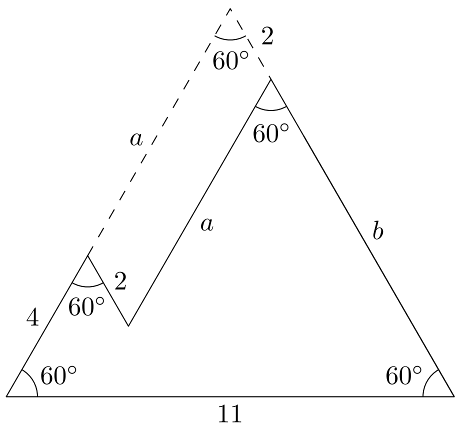
Solution. The sought number cannot be a single-digit number because its digit is either even or odd, and hence it is counted in one of those categories. Similarly, a two-digit number is not feasible as it would need to end with the digit 2, which is even. Assume that the number of digits is 3 therefore, the sum of even and odd digits is also 3. Hence, the possible numbers are 123, 213, and 303. After applying the transformation from the problem statement, all of these numbers end up as 123, including 123. Hence, 123 is the sought after solution.

Problem 9. In the following figure, there is a pentagon with several given angles and sidelengths. Determine $a + b$.



Result. 16

Solution. Let us extend the pentagon with a parallelogram to an equilateral triangle of side length $11 = 4 + a = 2 + b$ as in the picture.



We conclude that $a = 7$, $b = 9$, and $a + b = 16$.

Problem 10. In the Slovak folk song *Kopala studienku* ("She was digging a well"), a girl checks whether her well is equally deep and wide. By definition, a well is a right cylinder, whose height is the depth of the well and whose base diameter is the width of the well. She knows that she needs a week to dig a well of the desired width but only $\frac{1}{3}$ of its required depth, while Janko Matúška needs a week to dig a well that is sufficiently deep but half as wide as needed. The effort is proportional to the amount of earth removed. How many days will they need to dig a proper well together?

Result. 12

Solution. The time needed to dig a well is assumed to be proportional to its volume. Because it is a right circular cylinder, its volume is given by the formula $V = \frac{\pi}{4} \cdot D \cdot W^2$, where D stands for depth and W stands for width. Now we can reason that the girl needs 7 days to dig out $\frac{\pi}{4} \cdot \frac{D}{3} \cdot W^2 = \frac{1}{3}V$ of the earth, thus she needs 21 days to dig out the whole volume V of earth. Similarly, Janko needs 7 days to dig out $\frac{\pi}{4} \cdot D \cdot \left(\frac{W}{2}\right)^2 = \frac{1}{4}V$ of the earth, thus he needs 28 days to dig out the whole volume V of earth. Therefore, the girl can perform $\frac{1}{21}$ of the well in a day, and her companion can perform $\frac{1}{28}$ of the well in a day. Together, they can perform $\frac{1}{21} + \frac{1}{28} = \frac{1}{12}$ of the well in a day. Finally, we can conclude that they need 12 days to build a well together.

Problem 11. Calculate the number of all segments of length $\sqrt{5}$ connecting two square corners in a grid of 10×10 unit squares.

Result. 360

Solution. First we make the observation that every 2×1 rectangle contains exactly 2 diagonals of length $\sqrt{5}$. So our goal is to enumerate the number of 2×1 rectangles. Suppose that the rectangle is oriented vertically. Then we have 10 possibilities to place it in a column and 9 possibilities for a row. So we have 90 possibilities to place this rectangle in a 10×10 grid. We can also orient the rectangle horizontally with the same number of possible placements. In conclusion, we have two diagonals in each of $90 + 90 = 180$ rectangles, so we have 360 diagonals in total.

Problem 12. If M, A, T, and H are distinct non-zero digits such that the equation

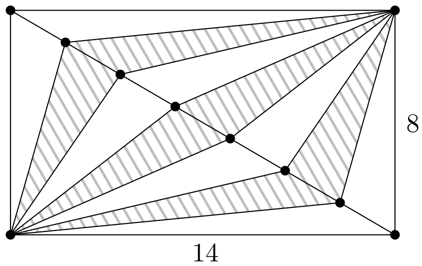
$$2024 + HAHA = MATH$$

holds, what is the largest possible value of the four-digit number $MATH$?

Result. 5963

Solution. Since $MATH$ and $HAHA$ have the same hundreds digit and the corresponding digit in 2024 is 0, we see that there is no carry when adding the tens digits on the left-hand side. Therefore, $TH = HA + 24$ and $M = H + 2$. However, adding A and 4 must produce a carry, for otherwise we would have $T = H + 2 = M$, and the digits are assumed to be distinct. This implies that $H = A + 4 - 10 = A - 6$, hence $M = A - 4$ and $T = A - 3$. From this, we easily get that $MATH$ is one of 3741, 4852, or 5963, the last value being the greatest.

Problem 13. In the zebra-rectangle with side lengths of 14 and 8, the diagonal is dissected into seven line segments of equal length. How large is the shaded area?



Result. 48

Solution. Since the altitudes from the two vertices to the diagonal have equal lengths for all triangles involved, the shaded area is exactly $\frac{3}{7}$ of the total area. Hence the solution is $\frac{3}{7} \cdot 8 \cdot 14 = 48$.

Problem 14. If a new girl joins the math club but 20% of boys leave it, the numbers of boys and girls would be equal. On the other hand, if one girl leaves the math club and next time the number of girls increases by 30%, the numbers of boys and girls would also be equal. How many children are there in the math club?

Result. 116

Solution. Let us denote the number of girls by g and the number of boys by b. The statement gives us the following system of equations:

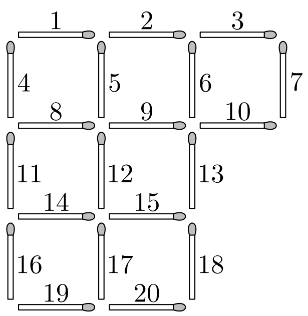
$$\begin{aligned} g + 1 &= \frac{4}{5}b, \\ \frac{13}{10}(g-1) &= b. \end{aligned}$$

Plugging in $b = \frac{5}{4}g + \frac{5}{4}$ from the first equation into the second equation, we get

$$\frac{5}{4}g + \frac{5}{4} = \frac{13}{10}g - \frac{13}{10},$$

which solves for $g = 51$, hence $b = \frac{13}{10} \cdot 50 = 65$. Therefore, the answer is $g + b = 51 + 65 = 116$.

Problem 15. The matchsticks in the picture form nine squares. Remove three of them so that there are exactly five squares left and each matchstick remains a part of some square. Find the maximal possible sum of numbers assigned to such a triplet of removed matches.



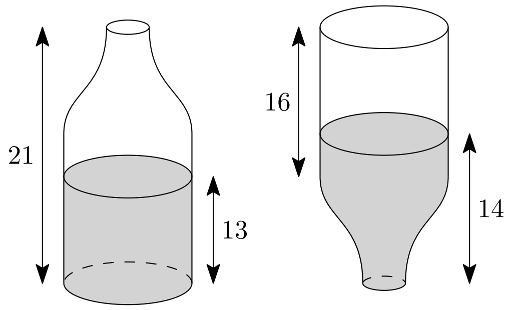
Result. 50

Solution. There are 7 squares of side length 1 and two squares of side length 2. To reduce the number of squares to 5, one has to break exactly 4 squares. To remove a square bounded by matchsticks $3, 7, 6, 10$, one needs to remove at least three of them. Hence, it is one of the squares that will survive. It is important to note that matchstick 6 cannot be removed as it would break this square and we cannot collect the remaining matchsticks from this square.

Removing matchsticks 11 or 13 breaks both large squares simultaneously. If that happens, then one square needs to be broken with two removals. Consider a case where a matchstick of 11 is drawn; then there is a possibility of taking pair off matchsticks 12, and 13 or the pair of 18, 20. On the other hand, if matchstick 13 was taken, then it could be either the pair of 1, 4, the pair of 11, 12 or the pair of 16, 19. Out of which, the removal of matchsticks 11, 18, and 20 has the highest sum of 49. If both large squares are intact, that means that the only matchsticks that can be removed are 5, 12, and 17 and that does not result in a valid configuration.

Since matchstick 6 cannot be taken and one large square needs to be broken, it is safe to assume that both matchsticks from one of the following pairs are removed: 18, 20 or the pair 16, 19, or the pair 1, 4. Such removal takes two squares at the same time, one large and one small. After that, there needs to be one matchstick that breaks exactly two squares. In the case of the pair 18, 20, it can be either matchstick 11 breaking one small and the large one or 12 breaking two small circles, leaving a total sum of 50. Matchstick 13 cannot be taken, as matchstick 15 would not be part of any square. In a similar manner, it can be assumed that breaking pair $16, 19$ alongside matchstick 13 results in a total sum of 48. Hence 50 is the sought answer.

Problem 16. Lukáš has a bottle of height 21. It consists of a cylinder of height 16 and an irregular shape at the bottleneck. Lukáš partly filled the bottle with water. He figured out that the water reached a height of 13. Then he turned the closed bottle upside down and noticed that the water reached a height of 14. Calculate the percentage of the volume of the bottle that was filled with water.

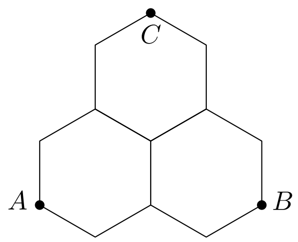


Result. 65

Solution. Let r denote the base radius. From the first configuration, the volume of water in the bottle is $13\pi r^2$. Similarly, from the second configuration, the volume of air in the bottle is $(21-14)\pi r^2 = 7\pi r^2$. Hence, the bottle has a total volume of $(13+7)\pi r^2 = 20\pi r^2$ and the desired percentage is

$$\frac{13\pi r^2}{20\pi r^2} = \frac{13}{20} = 65\,\%.$$

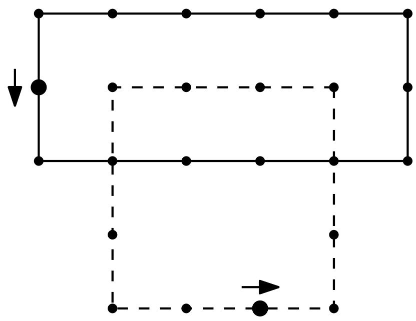
Problem 17. Ondra lives in Hexagonia, a city in which all streets are 1 km long sides of three regular hexagons. He wants to pick up his girlfriend first and then go to the cinema with her. On the picture, Ondra starts at point A, his girlfriend lives at point B and the cinema is located at point C. He does not want to walk any of the streets twice. What is the sum of the lengths of all possible paths he can take (in kilometers)?



Result. 28

Solution. There are four paths from A to B that do not pass through C. One of them allows two ways to get to C, one path has exactly one possibility to get to C and for the other two paths, it is not possible to get to C without walking a street twice. Altogether, there are three valid paths to the cinema via his girlfriend. Two of them have lengths of 10, the other one has a length of 8, hence the result is 28.

Problem 18. There are two guards patrolling along rectangular routes, as shown in the picture. They walk at a constant speed, passing from one marked checkpoint to its neighbor in one minute. After how many minutes will they meet for the first time?



Result. 44

Solution. Let guard A be the one who needs 14 minutes (rectangular path) and B the one who needs 12 minutes (square path) to complete one round. There are two possible meeting points for the guards. If they meet at the left one after a full rounds made by A and b full rounds made by B, then the condition

$$14a + 2 = 12b + 8$$

must hold. This equation simplifies to $7a = 6b + 3$, implying $7 \mid 6b + 3$. By trying $b \in \{0, 1, 2, \dots\}$ we see that $b = 3$ and $a = 3$ is the smallest solution. Similarly, the guards meet at the right point if

$$14a + 5 = 12b + 3$$

is satisfied. Hence, we get $7a = 6b - 1$. So we have $7 \mid 6b - 1$, which forces $b \geq 6$. Hence, they meet for the first time after $14 \cdot 3 + 2 = 44$ minutes.

Problem 19. Positive integers a, b, and c satisfy the equations

$$\sqrt{a^2 + b^2 - 172} = c,$$
$$\sqrt{c^2 + b^2 - 220} = a.$$

What is the largest possible value of their sum $a + b + c$?

Result. 26

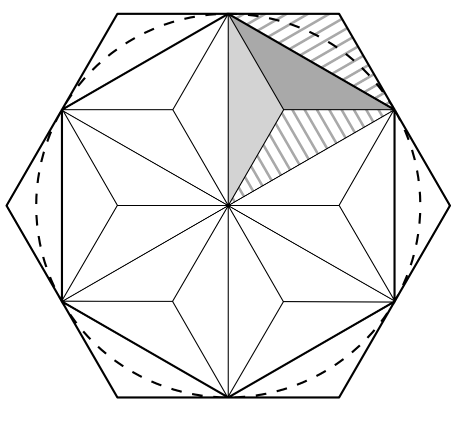
Solution. Square both equations and compute their sum to obtain $2b^2 = 392$. Since b has to be positive, $b = 14$ is the only solution. Plugging this value into the square of the first equation, we obtain $a^2 + 24 = c^2$ or $c^2 - a^2 = 24$. Put $d = c - a$; then d is an even number, since c and a are either both even or both odd.

The value of $c^2 - a^2 = (c - a)(c + a)$ can be bounded from below by $d(d + 2)$, which for $d \geq 6$ is at least 48, a number larger than 24. Hence, the only options for d are $d = 2$ and $d = 4$. In the former case, we obtain $a + c = 12$ with the solution $a = 5$ and $c = 7$. In the latter case, $a + c = 6$ with the solution $a = 1$ and $c = 5$. Therefore, the largest possible value of $a + b + c$ is $5 + 14 + 7 = 26$.

Problem 20. Take a circle, then inscribe and circumscribe two regular hexagons. What part of the area of the circumscribed hexagon is covered by the inscribed hexagon?

Result. $\frac{3}{4}$

Solution. Subdividing into congruent triangles as in the figure and counting leads to the answer $\frac{18}{24} = \frac{3}{4}$.



Alternative solution. Let r be the radius of the circle. Both hexagons can be subdivided into 6 equilateral triangles; for the inscribed hexagon, the height of each triangle is $\frac{1}{2}\sqrt{3}\,r$, while for the circumscribed one it is r. Therefore, the scaling factor for lengths is $k = \frac{1}{2}\sqrt{3}$, and consequently, the scaling factor for areas is $k^2 = \frac{3}{4}$.

Problem 21. We call the n-th birthday a *square birthday* if $n > 1$ and whenever a prime number p divides n, then also p^2 divides n. For example, $n = 8 = 2^3$ works, while $n = 56 = 8 \cdot 7$ does not. This year, Grandpa Jefo celebrated his 196th birthday. How many square birthdays has he lived through?

Result. 20

Solution. Any square birthday number needs to be composed of one or multiple factors of the form p^k where $k > 1$. All such factors less or equal to 196 are $S = \{4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196\}$. We can see that the product of at least two numbers from S greater than 27 either belongs to S or is larger than 196. From the numbers smaller or equal to 27, only $8 = 2^3$ and $27 = 3^3$ are not perfect squares. Since a product of perfect squares is a perfect square, it either means that the result would either belong to S or be larger than 196. Finally, the products using 8 or 27 and some other numbers from S that are smaller than 196 and not already in S are $27 \cdot 4 = 108$ and $8 \cdot 9 = 72$. It follows that there are $18 + 2 = 20$ such numbers.

Problem 22. The rip-off mathematical contest Mathematical Charge has been ongoing for 10 years. In year n, the number of questions in the contest was $n + 2$, all being numbered in the usual way from 1 to $n + 2$. For the 11-th edition of the contest, the organizers want to take one question from each of the previous years so that they can form a test of 10 questions numbered 1 to 10, using the already existing numbers. How many different tests could they create, assuming no two questions from the previous contests are identical?

Result. $3^8 \cdot 2 = 13122$

Solution. The organizers have three questions to choose from in the contest nr. 1. They can choose one of the $4 - 1 = 3$ possible questions from the second contest, as one question number is already occupied. It is easy to see that this pattern stays, namely that in the k-th contest there are already $k - 1$ questions unavailable due to (some) previous choices leaving three options, until contest nr. 9 which contains a question number 11 which is too large, thus leaving only two questions to choose. Finally, from the test nr. 10, there are questions nr. 11, and nr. 12, which cannot be chosen, leaving only one suitable question. Altogether, there are $3^8 \cdot 2 = 13122$ such tests.

Problem 23. Determine the smallest positive integer that has 1 as its first digit and the following property: When the digit 1 is relocated to the end of the number, the resulting number is three times the original number.

Here is an example of relocating the first digit: $174 \to 741$.

Result. 142857

Solution. Knowing that the last digit of the number is 1, one can try to reconstruct the number backwards as follows:

$$\begin{aligned} \dots x \cdot 3 &= \dots 1 \Rightarrow x = 7 \\ \dots y7 \cdot 3 &= \dots 71 \Rightarrow y = 5 \\ \dots z57 \cdot 3 &= \dots 571 \Rightarrow z = 8 \\ \dots t857 \cdot 3 &= \dots 8571 \Rightarrow t = 2 \\ \dots s2857 \cdot 3 &= \dots 28571 \Rightarrow s = 4 \\ \dots r42857 \cdot 3 &= \dots 428571 \Rightarrow r = 1. \end{aligned}$$

Indeed, $142857 \cdot 3 = 428571$ holds.

Alternative solution. Any positive integer starting with the digit 1 and at least two digits can be written as $10^k + a$ for some $k \geq 1$ and some k-digit number a. After relocating the digit 1 from the beginning to the end, the number changes to $10a + 1$. Hence, we want to solve the equation

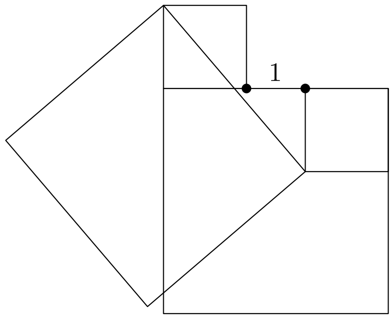
$$3 \cdot (10^k + a) = 10a + 1$$

for k and a; let us simplify it to

$$3 \cdot 10^k - 1 = 7a.$$

The number on the left-hand side is nothing else than a 2 followed by k nines. Take as many nines as needed and divide the number $2999\dots$ by 7 until there is no remainder. This process leads to $a = 42857$ and thus to the solution 142857.

Problem 24. The picture shows a configuration of two pairs of congruent squares (of positive sidelengths), with the two marked points having distance one. What is the sum of the areas of the four squares?

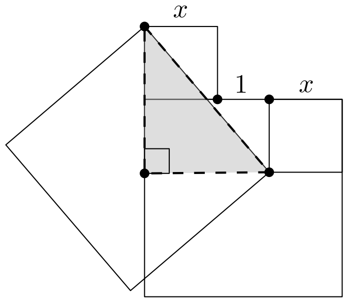


Result. 58

Solution. Denoting x the sidelength of the smaller squares and using the Pythagorean theorem for the shaded right triangle in the picture, we get

$$(2x)^2 + (1+x)^2 = (1+2x)^2.$$

This equation simplifies to $x^2 = 2x$, hence we get $x = 2$. The answer is then $2(2^2 + 5^2) = 58$.



Problem 25. Climber Christian is being lowered off the top of a vertical wall. This means that he is tied to one end of the rope, which goes through a fixed point on the top of the wall and then down to Lukas, who stands on the ground and lets the rope slip in a controlled way. The rope is elastic, and Christian's weight makes its loaded part (between him and Lukas) stretch by 20 %. The rope has a mark in the middle, and as Christian is being lowered off, he meets that mark at one third of the height of the wall above the ground. He is relieved, as this assures him that the rope is long enough, and he starts wondering how high the wall actually is. When he touches the ground and the rope has not gone slack yet, there are still 10 meters of loose rope left. Neglecting the heights of the people and lengths of the knots used, what is the height of the wall in meters?

Result. 18

Solution. Let us denote the length of the unstretched rope by l and the height of the wall by h. When the climber meets the mark, one half of the rope (stretched) covers twice the distance of the climber from the top, so we have

$$\frac{6}{5} \cdot \frac{l}{2} = 2 \cdot \frac{2h}{3}.$$

When the climber touches the ground, we similarly obtain

$$\frac{6}{5}(l - 10) = 2h,$$

which, after substituting $l = \frac{20h}{9}$ from the first equation, gets easily solved and yields $h = 18$.

Problem 26. A drawer contains n socks. When two socks are drawn randomly without replacement, the probability that both of them are black is $2/15$. What is the smallest possible value of n?

Result. 10

Solution. Let b denote the number of black socks. Then the probability that both socks are black is $\frac{b}{n} \cdot \frac{b-1}{n-1}$. Since this expression is equal to $\frac{2}{15}$, the following equation must hold:

$$15 \cdot b \cdot (b-1) = 2 \cdot n \cdot (n-1).$$

Since both 3 and 5 divide the left-hand side and both are coprime to 2, they must divide $n \cdot (n-1)$ on the right-hand side. Now start with small multiples of 3 and 5 for n to discover that $n = 6$ leads to $15 \cdot b \cdot (b-1) = 2 \cdot 6 \cdot 5 = 60$. However, $b \cdot (b-1) = 4$ cannot be fulfilled by any integer, therefore $n = 6$ is not a solution. If $n = 10$, then $b \cdot (b-1) = 12$ can be achieved by $b = 4$. Therefore, the answer for n is 10.

Problem 27. Find the largest integer satisfying the following conditions:

- it has exactly seven digits,
- no two digits are the same,
- it is a multiple of 11.

Result. 9876504

Solution. We will use the condition of divisibility by 11: a number is divisible by 11 if and only if the difference between its sum of digits on odd positions and its sum of digits on even positions is divisible by 11.

Among all numbers with a specified number of digits, seven in this case, the largest are the ones that start with the largest digits. Thus, let us start looking for a solution by choosing the digits from 9 downwards. After writing 98765, we see that the sum of the "odd" group is $9+7+5=21$ and the sum of the "even" group is $8+6=14$. The difference is 7 and we have to make it divisible by 11, using just two digits from the set $\{0,1,2,3,4\}$. The only way is to add 0 to the "even" group and 4 to the "odd" group, which produces the solution 9876504. Because all the other solutions would have to begin with a five-digit sequence smaller than 98765, this is indeed the largest one.

Alternative solution. Start with the largest number, 9876543, consisting of seven distinct digits. Observe either using the rule about division by 11 or doing long division that this number is not divisible by 11, but 9876537 is, and this is the largest multiple of 11 smaller than the number we started with. Since its digits are not distinct, we subtract 11 and check the digits of the newly obtained number with respect to distinctness. After some steps

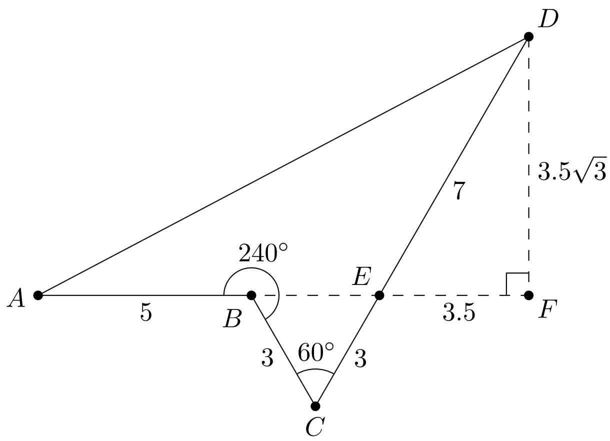
$$9876537 \longrightarrow 9876526 \longrightarrow 9876515 \longrightarrow 9876504$$

we obtain the sought-after number, 9876504.

Problem 28. Consider a quadrilateral $ABCD$ with side lengths $AB=5$, $BC=3$, and $CD=10$. The measure of the internal angle at B is $240°$, while the one at C is $60°$. Calculate the length AD.

Result. 13

Solution. Let us create an equilateral triangle BCE with E on the segment CD. Then AED is a triangle where $AE=8$, $ED=7$, and $\angle DEA=120°$. Therefore, by the Law of Cosines, $AD^2=8^2+7^2-2\cdot 7\cdot 8\cdot\cos 120°=169$, so $AD=13$.



Alternative solution without using the Law of Cosines. If the triangle AED is extended by half of an equilateral triangle of side length 7, we can use the Pythagorean theorem to get $AD^2=(5+3+3.5)^2+(3.5\cdot\sqrt{3})^2=169$.

Problem 29. How many ordered quadruples (a,b,c,d) of positive integers satisfy the equation

$$2024=(2+a)\cdot(0+b)\cdot(2+c)\cdot(4+d)?$$

Result. 18

Solution. First of all we need the prime factorization of 2024, which is

$$2024=2^3\cdot 11\cdot 23.$$

Since a, b, c, and d are positive integers, we have $2+a\geq 3$, $2+c\geq 3$, and $4+d\geq 5$. A factor 1 or a factor 2 on the right hand side may occur only once in $(0+b)$ and a factor 4 may appear only either in $(2+a)$ or $(2+c)$.

Since the product on the right-hand side of the given equation consists of four factors, there are four possible factorizations of 2024, with at most one factor being less than 4, namely

$$2024 = 1 \cdot 8 \cdot 11 \cdot 23 \quad \text{and} \quad 2024 = 1 \cdot 4 \cdot 22 \cdot 23 \quad \text{and} \quad 2024 = 1 \cdot 4 \cdot 11 \cdot 46 \quad \text{and} \quad 2024 = 2 \cdot 4 \cdot 11 \cdot 23.$$

For the first factorization, we have $b = 1$, and the remaining factors can be assigned to $a + 2$, $c + 2$, and $d + 4$ in any order, yielding 6 solution. In the second factorization, we have $b = 1$, then either $a + 2 = 4$ or $c + 2 = 4$. And in each of these cases, the remaining factors can be assigned in two ways, so there are in total 4 solutions for the second factorization. Analogously, there are 4 solutions for each of the third and fourth factorizations. Therefore, in total, there are 18 different solution quadruples:

factorization of 2024	solution			
	a	b	c	d
$2024 = 8 \cdot 1 \cdot 11 \cdot 23$	6	1	9	19
$2024 = 8 \cdot 1 \cdot 23 \cdot 11$	6	1	21	7
$2024 = 11 \cdot 1 \cdot 8 \cdot 23$	9	1	6	19
$2024 = 11 \cdot 1 \cdot 23 \cdot 8$	9	1	21	4
$2024 = 23 \cdot 1 \cdot 8 \cdot 11$	21	1	6	7
$2024 = 23 \cdot 1 \cdot 11 \cdot 8$	21	1	9	4
$2024 = 4 \cdot 2 \cdot 11 \cdot 23$	2	2	9	19
$2024 = 4 \cdot 2 \cdot 23 \cdot 11$	2	2	21	7
$2024 = 11 \cdot 2 \cdot 4 \cdot 23$	9	2	2	19
$2024 = 23 \cdot 2 \cdot 4 \cdot 11$	21	2	2	7
$2024 = 4 \cdot 1 \cdot 22 \cdot 23$	2	1	20	19
$2024 = 4 \cdot 1 \cdot 23 \cdot 22$	2	1	21	18
$2024 = 4 \cdot 1 \cdot 46 \cdot 11$	2	1	44	7
$2024 = 4 \cdot 1 \cdot 11 \cdot 46$	2	1	9	42
$2024 = 22 \cdot 1 \cdot 4 \cdot 23$	20	1	2	19
$2024 = 23 \cdot 1 \cdot 4 \cdot 22$	21	1	2	18
$2024 = 46 \cdot 1 \cdot 4 \cdot 11$	44	1	2	7
$2024 = 11 \cdot 1 \cdot 4 \cdot 46$	9	1	2	42

Problem 30. Let x and y be positive integers such that

$$2^x \cdot 3^y = \left(24^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{60}}\right) \cdot \left(24^{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{60}}\right)^2 \cdot \left(24^{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{60}}\right)^3 \cdots \left(24^{\frac{1}{60}}\right)^{59}.$$

Determine $x + y$.

Result. 3540

Solution. Considering $2^x \cdot 3^y = 24^k$, we have

$$\begin{aligned} k &= \frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) + \cdots + \left(\frac{1}{60} + \frac{2}{60} + \cdots + \frac{59}{60}\right) = \\ &= \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \cdots + \frac{59}{2} = \\ &= \frac{1}{2} \cdot \frac{(1+59) \cdot 59}{2} = \\ &= 15 \cdot 59. \end{aligned}$$

Then, $2^x \cdot 3^y = (2^3 \cdot 3^1)^{15 \cdot 59}$, which means that $x = 3 \cdot 15 \cdot 59 = 45 \cdot 59$ and $y = 15 \cdot 59$. Therefore, $x + y = 60 \cdot 59 = 3540$.

Problem 31. Annie loves apples, especially sequences of red and green apples of length 18 arranged in such a way that each dozen of consecutive apples contains at least seven green apples. How many such sequences containing at most eight green apples in total exist?

Result. 21

Solution. Let us focus just on the first and last dozen for the moment. If the middle semi-dozen of apples (i.e., apples 7–12) contains just green apples, both the first and last dozen are missing only one green apple, which is easily fixed by placing a green apple into both the first and last semi-dozens, so eight green apples are enough. If some of the middle-dozen-apples are red, one would need more green apples in total, as for each green apple removed from the middle semi-dozen one needs to add two apples (one into the first semi-dozen and the other one into the last semi-dozen). Therefore, 8 is the minimal amount of green apples needed, with six of them being placed in the middle, one in the beginning, and one towards the end.

However, the first and last green apples cannot be placed into their semi-dozens arbitrarily. In order to meet the condition for every dozen consecutive apples, the distance between the two green apples cannot exceed 12, e.g. if the first green apple is placed at position 2, the last can be placed either at position 13 or at position 14. Depending on the position of the first apple, the last apple thus has 1 to 6 possible positions. Adding these up gives $1+2+3+4+5+6 = 21$ possible sequences.

Problem 32. Fero has coins with values of 1 ct, 2 ct, and 5 ct. He has 33 of the 1 ct type, 106 of the 2 ct and 31 of the 5 ct type. He wants to split them into two piles with the same number of coins and the same value and give one of the piles to his sister. In how many ways could he do this? Coins of the same value are considered indistinguishable.

Result. 12

Solution. Let a, b, c be the number of 1 ct, 2 ct, and 5 ct coins, respectively, in the sister's pile. Then, we have the equations

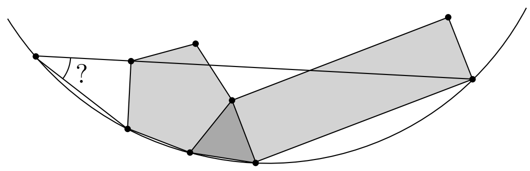
$$a + b + c = \frac{1}{2}(33 + 106 + 31) = 85,$$

and

$$a + 2b + 5c = \frac{1}{2}(33 + 2 \cdot 106 + 5 \cdot 31) = 200.$$

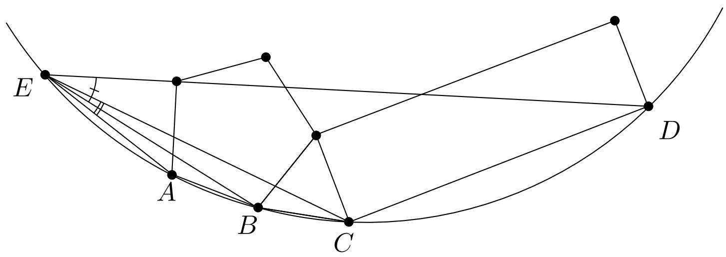
By subtracting the first equation from the second, we obtain $b + 4c = 115$. This equation has multiple solutions of the form $b = 115 - 4c$ for any given c. However, the constraints $0 < 115 - 4c < 106$ for b imply $c \in \{3, 4, \dots, 28\}$. But not all solutions have a valid amount of 1 ct coins. Therefore, we have to add the condition $0 \leq 85 - (115 - 4c + c) = -30 + 3c \leq 33$ for c. We can see that only values of c from the set $\{10, 11, \dots 21\}$ lead to a valid solution. Hence, there are 12 possible ways.

Problem 33. In the figure, an equilateral triangle, a regular pentagon, and a rectangle are such that some of their vertices are on a circle (only a part of which is shown). Find the measure of the marked angle in degrees.



Result. 36°

Solution. Let us recall that, given a circle ω, the angle under which a segment XY with endpoints on ω is visible from another point Z on ω is given only by on which of the two arcs (in which X and Y split the ω) it lies and the sum of these two values equals 180°.



In our situation, labelling some of the points as in the picture above and using the well-known measures of internal angles of regular pentagon and triangle, we have

$$\angle AEC = 180° - \angle ABC = 180° - 60° - 108° = 12°.$$

Similarly, we have

$$\angle BED = 180° - \angle BCD = 180° - 60° - 90° = 30°.$$

Finally, since ABC is an isosceles triangle, we have

$$\angle BEC = \angle BAC = \frac{180° - 108° - 60°}{2} = 6°$$

and hence, because of how these three angles at E overlap, we compute

$$\angle AED = 12° + 30° - 6° = 36°.$$

Problem 34. In how many ways can we place 9 rooks on a 4×4 chessboard such that each rook is attacked by some other rook? Two rooks attack each other if they are in the same row or column.

Result. 11296

Solution. Let us count the number of configurations where at least one rook is not attacked by any other rook. Such a rook needs to be alone in the row and the column at the same time, which means that there is at most one such rook. It can be placed at any square in $4 \cdot 4 = 16$ ways. Removing the corresponding row and column leaves nine squares where the remaining eight rooks must fit. The empty square can be chosen in 9 ways. Altogether, there are $16 \cdot 9 = 144$ ways of doing that. The total number of ways to choose nine squares from sixteen squares equals $\binom{16}{9} = 11440$, and hence the desired result is $11440 - 144 = 11296$.

Problem 35. Find the largest positive integer N that is not a prime number and all its divisors except for N itself are smaller than 100.

Result. 9409

Solution. Since N is not a prime number, it is either 1 or there is a prime number $p < N$ that divides N. The requirement $p < 100$ leads to

$$p \leq 97.$$

Observe that $N = 97^2 = 9409$ satisfies the conditions.

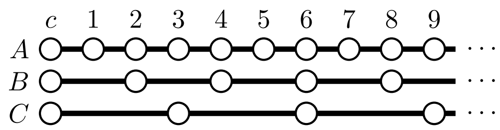
Now suppose there is a number $N' > 9409$ that also meets the given conditions. If $p \leq 97$ is a prime number that divides N', the quotient $\frac{N'}{p}$ is an integer greater than 97. This yields $\frac{N'}{p} \in \{98, 99\}$, since any divisor of N' has to be smaller than 100. But then N' is divisible by $k \in \{2, 3\}$, and the given condition leads to

$$N' = k \cdot \frac{N'}{k} \leq 3 \cdot 99 < 97^2,$$

which is a contradiction.

Therefore, $N = 9409$ is the sought number.

Problem 36. In Line City, there are three bus lines, a central station, and bus stops numbered by positive integers $1, 2, 3, \ldots$ All three lines start at the central station, denoted as c in the figure, and then pass all the stops in increasing order. Line A stops at all of them (numbers $1, 2, 3, \ldots$), line B stops at every second one (numbers $2, 4, 6, \ldots$), and line C stops only at every third stop (numbers $3, 6, 9, \ldots$). Danko starts his journey at the central station, picks a bus and aims to travel to stop nr. 17. At every station where his current bus stops, he can either get off the bus and take another one or continue on the same bus to its next stop. In how many ways can Danko reach his final destination if journeys differing only in waiting times are considered to be identical?



Result. 845

Solution. Denote a bus stop s_0 if all three bus lines stop, and denote s_k the k-th bus stop after s_0 by the bus lane A. Now calculate in how many ways Danko can reach the bus stop s_6 from the bus stop s_0.

1. Danko can reach the bus stop s_1 in exactly one way, and that is by the bus lane A.
2. There are two ways to get to the bus stop s_2, and that is either by bus lane A from the bus stop s_1 or by bus lane B from the bus stop s_0.
3. To reach the bus stop s_3, Danko either took the bus lane C from the bus stop s_0 or the bus lane A from the bus stop s_2, which can be reached by 2 ways; hence, there are 3 possible ways.
4. To reach the bus stop s_4, it is either lane A from stop s_3 or lane B from stop s_2 hence, there are $3 + 2 = 5$ ways.
5. To reach the bus stop s_5, there is only one way, and that is by lane A from stop s_4 hence, there are 5 ways.
6. Finally, to reach the bus stop s_6, it is either by the bus lane A from the bus stop s_5, or by the bus lane B from the bus stop s_4, or by the bus lane C from the bus stop s_3 hence, there are $3 + 5 + 5 = 13$ ways.

Bus stops where all lines stop are the central station c, the bus stop nr. 6 and the bus stop nr. 12. Since any bus stop where all bus lines stop can act as the bus stop s_0 there are 13 ways for Danko to reach the bus stop nr. 6 from the central station c and the bus stop nr. 12 from the bus stop nr. 6. It can be deduced that to reach the 17-th stop from stop nr. 12 is the same as to reach the bus stop s_5 from the s_0 and hence there are 5 ways. Together, there are $5 \cdot 13 \cdot 13 = 845$ ways for Danko to reach the stop nr. 17.

Alternative solution. Let us refer to the stops by their numbers, setting $c = 0$. Every stop s is reachable via line A, so every journey to the preceding stop $s-1$ can be extended to stop s via line A. If line B stops at s, then the journeys can be extended from stop $s-2$ to s via line B. A similar fact holds for stop s where line C stops. Therefore, if we denote by $J(s)$ the number of ways Danko can reach stop s, then for $s \geq 1$ we obtain

$$\begin{aligned} J(s) = J(s-1) & \\ + J(s-2) & \text{ if } s \text{ is divisible by } 2 \\ + J(s-3) & \text{ if } s \text{ is divisible by } 3. \end{aligned}$$

Since the central stop is "reachable" in only one way, we have $J(0) = 1$ and we can determine $J(17)$ using the given recurrence; the arrows under the table show which values are added to give the number in each cell.

s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$J(s)$	1	1	2	3	5	5	13	13	26	39	65	65	169	169	338	507	845	845

Problem 37. By $\lfloor x \rfloor$ we denote the largest integer that is not greater than the real number x. Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_1 = \sqrt{3}$ and for each $n \geq 1$, we have

$$a_{n+1} = \lfloor a_n \rfloor + \frac{1}{a_n - \lfloor a_n \rfloor}.$$

What is the value of a_{2024}?

Result. $3034 + \frac{\sqrt{3}+1}{2} = 3035 + \frac{\sqrt{3}-1}{2}$

Solution. Note that a_1 has the decimal part $a_1 - \lfloor a_1 \rfloor = \sqrt{3} - 1$. Hence, it can be written as $a_1 = 1 + \sqrt{3} - 1$. We calculate the first few terms

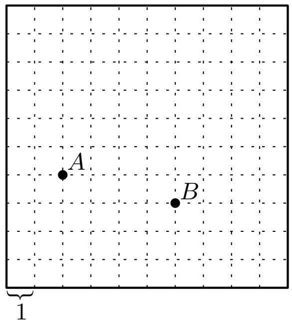
$$\begin{aligned} a_2 &= 1 + \frac{1}{\sqrt{3}-1} = 1 + \frac{\sqrt{3}+1}{2} = 2 + \frac{\sqrt{3}-1}{2}, \\ a_3 &= 2 + \frac{2}{\sqrt{3}-1} = 2 + \frac{2\sqrt{3}+2}{2} = 2 + \sqrt{3} + 1 = 3 + 1 + \sqrt{3} - 1, \\ a_4 &= 4 + \frac{1}{\sqrt{3}-1} = 4 + \frac{\sqrt{3}+1}{2} = 3 + 2 + \frac{\sqrt{3}-1}{2}. \end{aligned}$$

Observe that terms a_1 and a_3 have the same decimal part $\sqrt{3} - 1$ and the difference $a_3 - a_1 = 3$. The same reasoning applies to the terms a_2 and a_4, which have the same decimal part $\frac{\sqrt{3}-1}{2}$ and difference $a_4 - a_2 = 3$. This leads us to the hypothesis $a_{2k+1} = 3k + 1 + \sqrt{3} - 1$ and $a_{2k+2} = 3k + 2 + \frac{\sqrt{3}-1}{2}$, where $k = 0, 1, \ldots$. The validity for all k can be proven by induction; it is clear for $k = 1$ and $k = 2$. For the rest, it suffices to plug the formulas into the definition $a_{n+1} = \lfloor a_n \rfloor + \frac{1}{a_n - \lfloor a_n \rfloor}$. Observe that

$$\begin{aligned} a_{2k+2} &= \lfloor a_{2k+1} \rfloor + \frac{1}{a_{2k+1} - \lfloor a_{2k+1} \rfloor} = 3k + 1 + \frac{1}{\sqrt{3}-1} = 3k + 1 + \frac{\sqrt{3}+1}{2} = 3k + 2 + \frac{\sqrt{3}-1}{2}, \\ a_{2\cdot(k+1)+1} &= \lfloor a_{2k+2} \rfloor + \frac{1}{a_{2k+2} - \lfloor a_{2k+2} \rfloor} = 3k + 2 + \frac{2}{\sqrt{3}-1} = 3k + 2 + \frac{2 \cdot (\sqrt{3}+1)}{2} = 3 \cdot (k+1) + 1 + \sqrt{3} - 1. \end{aligned}$$

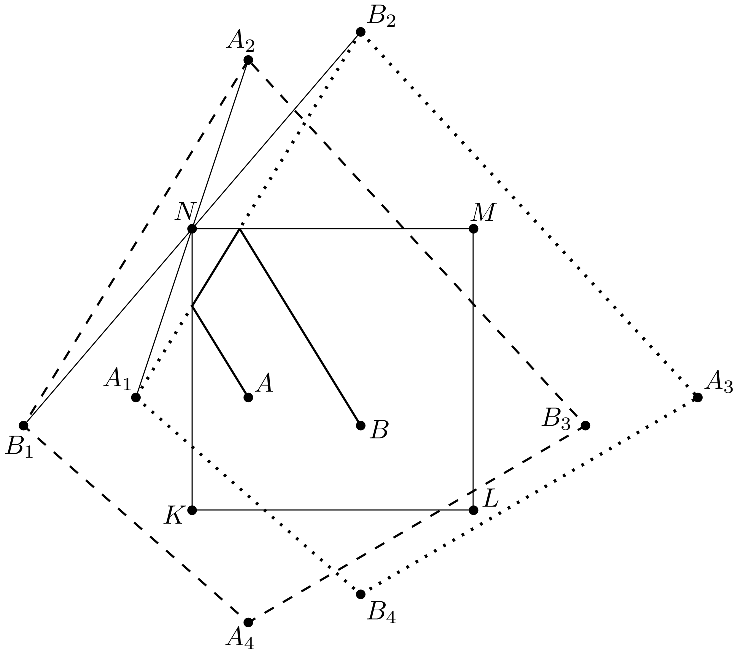
Hence, $a_{2024} = 3034 + \frac{\sqrt{3}+1}{2} = 3035 + \frac{\sqrt{3}-1}{2}$.

Problem 38. There is a square billiard table 10×10 with two balls, as shown in the picture. Each ball is dimensionless (a point), always moves straight, and when it hits a wall, it bounces off at the same angle. Consider all the paths, in which ball A bounces off exactly two walls before hitting ball B and compute the sum of squares of lengths of these paths.



Result. 2520

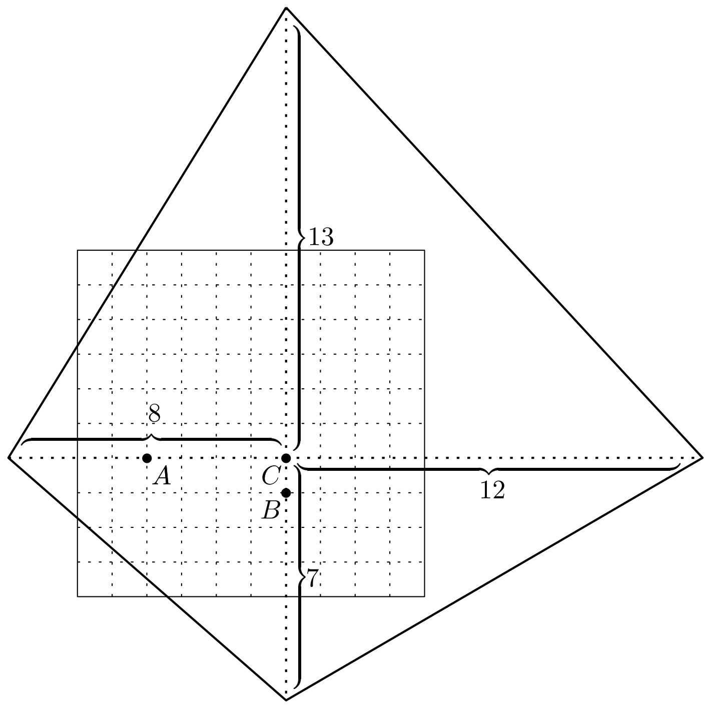
Solution. Let us first note that $A = [2, 4]$ and $B = [6, 3]$ with respect to the lower left corner of the square. Reflect A and B across the sides of the square and label everything according to the picture below.



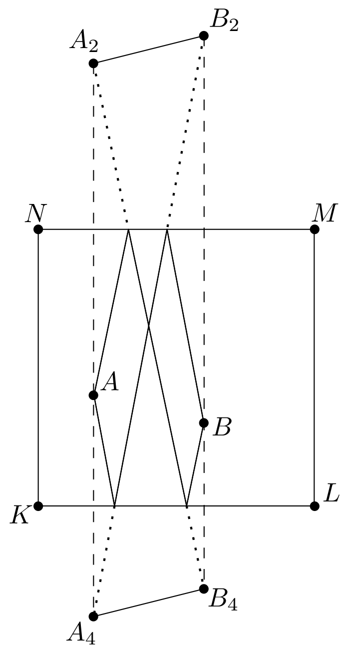
Consider the trajectory from A to B that bounces off KN and then MN and reflect its parts adjacent to the points A (resp. B) across KN (resp. MN). Due to the reflection angle rule, we obtain precisely the segment A_1B_2. (In the sequel, we will say that the trajectory was *straightened* into A_1B_2). Note that this is the only admissible trajectory that bounces off precisely these two sides of the square. Indeed, straightening a possible trajectory $A \to MN \to KN \to B$ results in the segment A_2B_1, which does not intersect the square $KLMN$. This is due to the properties of reflection implying that N is the midpoint of both A_1A_2 and B_1B_2. Therefore, such a trajectory is not possible.

Similarly, all the desired trajectories bouncing off two adjacent sides of the square $KLMN$ straighten into one side of the quadrilateral $A_1B_2A_3B_4$ or its shifted copy $B_1A_2B_3A_4$. From the corresponding congruent sides, exactly one is used because the other side is in the invalid configuration. It follows that these trajectories contribute to the sum of lengths squared by the sum of squares of the lengths of sides of $A_1B_2A_3B_4$. Using Pythagoras' theorem and the fact that its diagonals are perpendicular and intersect at point $C = [6, 4]$ (see the picture below), we calculate that

$$2(8^2 + 13^2 + 12^2 + 7^2) = 852.$$



Now it remains to consider the trajectories bouncing off two opposite sides of the square $KLMN$, such as the two shown below.



In this case, both orders are possible, resulting in a contribution equal to the sum of squares of the diagonals of parallelograms $A_2B_2B_4A_4$ and $A_1A_3B_3B_1$. Using the fact that the sum of squares of diagonals in a parallelogram equals the sum of squares of its sidelengths, we obtain

$$2(20^2 + 1^2 + 4^2) = 834,$$

for both of the parallelograms. Hence, the total sum is

$$852 + 2 \cdot 834 = 2520.$$

Problem 39. Let $x \parallel y$ denote a concatenation of two positive integers, that is, a number obtained by first writing out the digits of x as they appear in x and then doing the same for y; for example, $3 \parallel 4 = 34$, $24 \parallel 5 = 245$, and $20 \parallel 24 = 2024$. A positive integer n is called *threevisible* if there exist three pairwise distinct positive integers (without leading zeros) a, b, and c such that $n = a \parallel b \parallel c$ and a divides b and b divides c. What is the largest threevisible five-digit number?

Result. 94590

Solution. Note that since all numbers a, b, and c are distinct, and from the divisibility condition, it must hold that $2 \cdot a \le b$ and $2 \cdot b \le c$. Let $s(k)$ denote the number of digits of k. From the divisibility conditions, it follows that digit counts $s(a)$, $s(b)$ and $s(c)$ must be non-decreasing. Therefore, there are only two cases:

1. Digit counts satisfy $s(a) = 1$, $s(b) = 1$, and $s(c) = 3$. Then the number a is at most $4 < \frac{9}{2}$. This leads to the result $a = 4$, $b = 8$, and $c = 992$.

2. Digit counts satisfy $s(a) = 1$, $s(b) = 2$, and $s(c) = 2$. Then the number b is at most $49 < \frac{99}{2}$. To maximize the number $a \parallel b \parallel c$, assume that the first digit is 9. Then the maximal possible b is $b = 45$ and hence $c = 90$. Considering any smaller a leads to a smaller result.

Therefore, the maximal possible number is 94590.

Problem 40. By writing S_x, a string of digits with a subscript x (some positive integer larger than all used digits), we indicate that base x should be used to interpret the string. For example $242_7 = 2 \cdot 7^2 + 4 \cdot 7 + 2 = 128_{10} = 10000000_2$. Find the sum of all integers $x > 5$ for which the statement

$$2024_x \text{ is divisible by } 15_x$$

holds true.

Result. 471

Solution. We look for x such that the fraction $\frac{2x^3+2x+4}{x+5}$ is an integer. Since

$$\frac{2x^3 + 2x + 4}{x + 5} = 2x^2 - 10x + 52 - \frac{256}{x + 5},$$

it is sufficient for $x + 5$ to divide $256 = 2^8$. Since $x > 5$, we search for divisors that are at least 10. All such divisors are 16, 32, 64, 128, and 256. Therefore, the sought-after solution is the sum

$$\sum_{i=4}^{8} (2^i - 5) = 2^9 - 2^4 - 25 = 512 - 16 - 25 = 471.$$

Problem 41. We have two boxes: the first one contains five perfect light bulbs and nine faulty ones, whereas the second one contains nine perfect and five faulty bulbs. The perfect bulbs work always, whereas the faulty ones only with a probability p (where $0 < p < 1$), which is the same for all of them. Find the value of p, for which the following events have the same probability:

1. A randomly chosen bulb from the first box works.

2. Two randomly chosen bulbs from the second box both work.

Result. 7/20

Solution. Let us do straightforward combinatorics: The probability of the first event is

$$P_1 = \frac{1}{14}(5 + 9p),$$

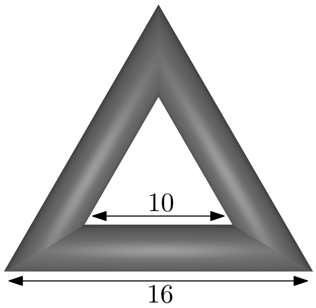
while for the second event we obtain

$$P_2 = \frac{1}{\binom{14}{2}} \left(\binom{9}{2} + 9 \cdot 5p + \binom{5}{2} p^2 \right).$$

Now our goal is to solve $P_1 = P_2$, which is a quadratic equation solvable via the usual approaches. However, we may realize that $p = 1$ is surely a solution, hence we may easily find the other one using Vièta's formulas: Recall that a quadratic equation $a \cdot (x - r_1) \cdot (x - r_2) = 0$ has the constant term $a \cdot r_1 \cdot r_2$, where r_1, r_2 are the roots and a is the coefficient of the quadratic term x^2. Therefore the solution can be found as

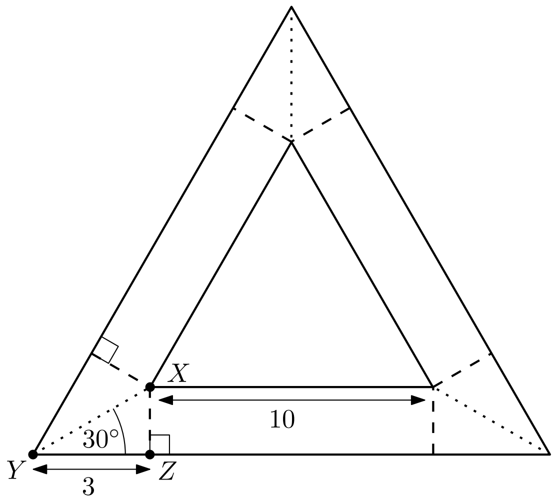
$$\frac{\frac{\binom{9}{2}}{\binom{14}{2}} - \frac{5}{14}}{\frac{\binom{5}{2}}{\binom{14}{2}}} = \frac{7}{20}.$$

Problem 42. Determine the volume of the body shown below, formed by three identically trimmed cylindrical tubes. The axes of the cylinders meet at the vertices of an equilateral triangle. The side lengths of the inmost and outmost contours (also equilateral triangles) are given.



Result. $\frac{117\pi}{4}$

Solution. Let us look at the plane containing the inmost and outmost contours and draw the segment XZ perpendicular to YZ as in the picture.



The symmetry of the involved equilateral triangles gives $|YZ| = 3$ and $|\sphericalangle ZYX| = 30°$, and thus $|XZ| = \sqrt{3}$. Cutting the full body along the planes indicated in the picture above by dotted and dashed lines splits it into three cylinders of radius $\frac{|XZ|}{2} = \frac{\sqrt{3}}{2}$ and height 10 and six pieces, which can be rearranged to form three cylinders of the same radius and height 3. The sought volume is thus

$$V = \pi \left(\frac{\sqrt{3}}{2}\right)^2 (3 \cdot 10 + 3 \cdot 3) = \frac{117\pi}{4}.$$

Problem 43. Ten pairwise distinct positive integers are written in a row so that

- the sum of any two consecutive numbers is divisible by 3,
- the sum of any three consecutive numbers is divisible by 2.

What is the smallest possible sum of the ten numbers?

Result. 78

Solution. The optimal construction is $2, 1, 5, 4, 11, 7, 8, 13, 17, 10$ with the sum of 78.

If there is a number divisible by 3, then also its neighbours are divisible by 3, etc., so all numbers are divisible by three. Therefore, the minimum possible sum is $3 \cdot (1 + 2 + \cdots + 10) = 3 \cdot \frac{10 \cdot 11}{2} = 165 > 78$, and it cannot be optimal.

Now, since the sum of three consecutive numbers should be divisible by 2, we have 2 options for each triplet: either there are three even numbers or there are two odd and one even number. Consider that there is a triplet x_i, x_{i+1}, x_{i+2} with three even numbers. Then the triplet x_{i-1}, x_i, x_{i+1} cannot contain two odd numbers hence, x_{i-1} is also even. Therefore, all ten numbers are even. The minimum possible sum is $2 \cdot \frac{10 \cdot 11}{2} = 110 > 78$, thus it cannot be optimal.

Therefore, in each triplet, there are two odd (O) numbers, and the third one is even (E). There are three possible configurations:

- OEOOEOOEOO – Summing 7 smallest odd and 3 smallest even numbers that are not divisible by 3 leads to the minimum possible sum is $1+5+7+11+13+17+19+2+4+8 = 87 > 78$, thus it cannot be optimal.
- OOEOOEOOEO – This configuration is symmetrical to the previous one.
- EOOEOOEOOE – Summing 6 smallest odd and 4 smallest even numbers that are not divisible by 3, leads to the minimum possible sum is $1+5+7+11+13+17+2+4+8+10 = 78$, which is the desired result.

Problem 44. Sophia is playing around with fractions. She wants to determine positive integers a, b fulfilling

$$\frac{2020}{2024} < \frac{a^2}{b} < \frac{999}{1000}$$

such that $a+b$ is minimal. Do the same as Sophia and give this minimal sum $a+b$ as the result.

Result. 553

Solution. The given inequality is equivalent to

$$\frac{1000}{999} < \frac{b}{a^2} < \frac{2024}{2020}.$$

Therefore, Sophia has to choose a as the smallest positive integer for which there is a positive integer b satisfying

$$\frac{1000}{999} \cdot a^2 < b < \frac{2024}{2020} \cdot a^2 \quad \Longleftrightarrow \quad a^2 + \frac{1}{999} \cdot a^2 < b < a^2 + \frac{4}{2020} \cdot a^2.$$

For $a < 32$, she gets $a^2 < a^2 + \frac{a^2}{999} < a^2 + 1$. If there is an $a < 32$ such that $\frac{4a^2}{2020} > 1$, she can take the minimal a fulfilling this inequality. Now

$$\frac{4 \cdot 22^2}{2020} = \frac{44^2}{2020} = \frac{1936}{2020} < 1 \quad \text{and} \quad \frac{4 \cdot 23^2}{2020} = \frac{46^2}{2020} = \frac{2116}{2020} > 1.$$

Therefore, $a = 23$ and $b = a^2 + 1 = 530$ fulfill the problem statement hence, the value asked for is $a+b = 23+530 = 553$.

Problem 45. The floor of a tent has the shape of a triangle with side lengths of 1.3, 2, and 2.1 meters. The manufacturer wants to advertise that a person of height h can lie there *arbitrarily* in the sense that every point of the floor belongs to a possible sleeping position (a segment of length at least h contained in the triangle). How much (in meters) can h be at most?

Result. $\frac{126}{65}$

Solution. We claim that the longest segment, which can be put inside an acute triangle (which ours clearly is) through its arbitrary point, is the longest altitude. By drawing all segments from one vertex to the opposite side, we cover the whole triangle, and the shortest segment we drew is the corresponding altitude (an acute triangle contains all its altitudes). It remains to be shown that there is no satisfying segment that is longer. Take a view at the foot of the longest altitude. If the corresponding side is shorter than the altitude, there cannot be a longer satisfying segment (since all segments containing the foot are of a length at most the maximum of the length of the altitude and the length of the corresponding side). The basic formula for the area of a triangle shows that the longest altitude belongs to the shortest side, which in our case is 1.3. Hence, if the corresponding altitude is greater than 1.3, we are done.

There are many ways to compute the length of the corresponding altitude. One would be using Heron's formula to compute the area and then dividing by the half of the side. We show it in a more elementary way. We scale all values by 10 (i.e., compute in decimeters instead of meters). Denote by x, $13-x$ the lengths at which the considered altitude intersects the side of length 13. Then, from Pythagoras' theorem, we have

$$\begin{aligned} 20^2 - x^2 &= 21^2 - (13-x)^2, \\ 26x &= 128, \\ x &= \frac{64}{13}. \end{aligned}$$

Hence, the altitude equals

$$\begin{aligned} h &= \sqrt{20^2 - \left(\frac{64}{13}\right)^2} \\ &= \frac{4}{13}\sqrt{25 \cdot 169 - 256} \\ &= \frac{4}{13}\sqrt{9 \cdot 9 \cdot 49} \\ &= \frac{252}{13}. \end{aligned}$$

Since $\frac{252}{13} > 13$, the altitude is longer than the side, as desired. The result in meters is thus $\frac{252}{130} = \frac{126}{65}$.

Problem 46. Find the largest positive integer q such that for any positive integer $n \geq 55$, the number q divides the product

$$n(n+4)(n-23)(n-54)(n+63).$$

Result. 40

Solution. Denote the product as A. Taking A modulo 5, we see that the factors are n, $n+4$, $n+2$, $n+1$, and $n+3$, respectively. Since they are distinct, at least one of them is $0 \pmod 5$, that is $5 \mid A$. If n is even, then there are three even factors in A, so $8 \mid A$. If n is odd, there are two even factors, $n-23$ and $n+63$, whose difference is 86. Furthermore, $86 \equiv 2 \pmod 4$, so exactly one factor is a multiple of 4, meaning $8 \mid A$. Together, this shows $40 \mid A$.

Taking $n = 59$, one gets that the largest powers of 2 and 5 dividing A are 8 and 5, respectively. Taking $n = 55$, we get $3 \nmid A$. Finally, for any prime $p > 5$, the factors of A occupy at most $5 < p$ residue classes mod p, so it is always possible to choose n such that $p \nmid A$.

Therefore the result is $q = 40$.

Problem 47. Adam, Bea, Charles, Daniel, and Erik are attending two courses. Adam and Bea attend only the first course, Charles and Erik attend only the other one, while Daniel attends both. Roberta knows that each course is attended by three students, but not by which three. So she asks everyone to point a finger randomly at a classmate from any of their courses (i.e., Daniel will choose each of the other four people with probability $\frac{1}{4}$, etc.). What is the probability that it will be possible for Roberta to figure out that Daniel is the one attending both courses?

Result. $\frac{3}{4}$

Solution. If one student is pointing a finger at the other student, we will say that there is a *connection* between them.

Roberta is able to identify Daniel if and only if at least one person from each course has a *connection* with Daniel. If Daniel has more than two *connections*, it is obvious, since no other person can have such many *connections*. Otherwise, Daniel has exactly one *connection* in each of the two courses.

Without loss of generality, assume that there are *connections* between Adam – Daniel and Charles – Daniel. Since Bea has no *connection* with Daniel, she is necessarily pointing at Adam. Analogously, Erik is pointing at Charles. Therefore, we have a path of *connections* Bea – Adam – Daniel – Charles – Erik, and since only possible *connection* paths of length 4 with all 5 students have Daniel in the middle, we are done.

On the other hand, if there is no *connection* of Daniel with one course, we can assume he is attending only the other course (since we do not have information about connection with the first one) and therefore indistinguishable from his classmates in that course.

Now we can compute the resulting probability.

1. Suppose that Adam and Bea are pointing at each other, while Charles and Erik are not pointing at each other. The probability of the first event is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. In the second event at least one from Charles and Erik must be pointing at Daniel hence, the probability $\left(1 - \frac{1}{4}\right)$, and finally, Daniel must be pointing at one from Adam and Bea $\left(\frac{2}{4}\right)$. An analogous situation is when Charles and Erik are pointing at each other and Adam and Bea are not.

2. Otherwise, at least one from Adam and Bea is pointing at Daniel $\left(1 - \frac{1}{4}\right)$; the same goes for Charles and Erik $\left(1 - \frac{1}{4}\right)$ and it is irrelevant where Daniel is pointing.

Summing up together, we get

$$\frac{1}{4} \cdot \left(1 - \frac{1}{4}\right) \cdot \frac{2}{4} + \frac{1}{4} \cdot \left(1 - \frac{1}{4}\right) \cdot \frac{2}{4} + \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{4}\right) = \frac{3}{32} + \frac{3}{32} + \frac{9}{16} = \frac{3}{4}.$$

Problem 48. Function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies

1. $f(x) = x^2$ for all $0 \leq x < 1$ and

2. $f(x+1) = f(x) + x + 1$ for all non-negative real x.

Find all values of x such that $f(x) = 482$.

Result. $15 + 11 \cdot \sqrt{2} = 15 + \sqrt{242}$

Solution. Let $\{x\}$ denote the fractional part of x. Assuming $\lfloor x\rfloor \geq 1$, we compute using the given conditions that

$$\begin{aligned} f(x) &= f(\lfloor x\rfloor + \{x\}) \\ &= \lfloor x\rfloor + \{x\} + f(\lfloor x\rfloor - 1 + \{x\}) = \cdots \\ &= \sum_{i=1}^{\lfloor x\rfloor} i + \lfloor x\rfloor \cdot \{x\} + f(\{x\}) \\ &= \frac{\lfloor x\rfloor \cdot (\lfloor x\rfloor + 1)}{2} + \lfloor x\rfloor \cdot \{x\} + \{x\}^2. \end{aligned}$$

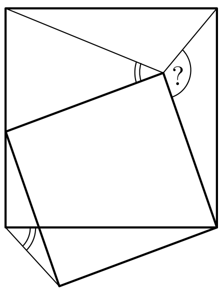
Note that the obtained formula holds for $\lfloor x\rfloor = 0$ as well.

Now we show that f is strictly increasing. Let us fix any integer $n \geq 0$. For $x, y \in [n, n+1)$ such that $x < y$, it is obvious from the definition that $f(x) < f(y)$. On the other hand, for all $x \in [n, n+1)$, the condition $f(x) < f(n+1)$ holds because

$$\begin{aligned} f(x) &= \frac{\lfloor x\rfloor \cdot (\lfloor x\rfloor + 1)}{2} + \lfloor x\rfloor \cdot \{x\} + \{x\}^2 \\ &< \frac{\lfloor x\rfloor \cdot (\lfloor x\rfloor + 1)}{2} + \lfloor x\rfloor + 1 \\ &= \frac{(\lfloor x\rfloor + 2) \cdot (\lfloor x\rfloor + 1)}{2} \\ &= \frac{(n+2) \cdot (n+1)}{2} \\ &= f(n+1). \end{aligned}$$

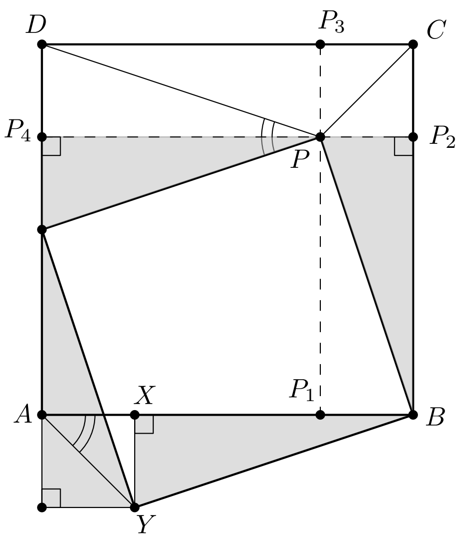
Therefore, there is at most one solution x to the given equation $f(x) = 482$. The greatest integer n such that $\frac{n^2+n}{2} \leq 482$ is 30 and hence $\lfloor x\rfloor = 30$. Now $482 = f(x) = \frac{\lfloor x\rfloor \cdot (\lfloor x\rfloor + 1)}{2} + \lfloor x\rfloor \cdot \{x\} + \{x\}^2 = 15 \cdot 31 + 30 \cdot \{x\} + \{x\}^2$ gives a quadratic equation for $\{x\}$ which has unique solution $\{x\} = -15 + \sqrt{242}$ belonging to $[0, 1)$. Hence $x = 30 - 15 + \sqrt{242} = 15 + 11\sqrt{2}$.

Problem 49. In the picture, there are two squares and a marked pair of equal angles. Determine the size of the missing angle in degrees.



Result. 112.5

Solution. Let us draw the perpendicular projections of the vertex adjacent to the sought angle on the sides of the big square and label all the points according to the picture.



It is easy to see that the four shaded triangles are congruent. Indeed, they are all right triangles with hypotenuse identical to one side of the smaller square and one angle equal to α given by how much the two squares are rotated with respect to each other. It follows that P_4PP_3D is a rectangle formed by another two copies of the shaded triangles and hence the angle marked by two stripes in the statement then equals 2α. Triangles AXY and PP_2C are right and isosceles (again because of the congruency of the shaded triangles) and hence $2\alpha = 45°$ and the sought angle can be computed as

$$90° - \alpha + 45° = 112.5°.$$

Problem 50. Ondra got bored of traditional operations like addition and multiplication. So he made up his own operation, *starification.* This operation, denoted by $a \star b$, is defined on real numbers and has the following properties:

1. $(a + b) \star c = (a \star c) + (b \star c)$,
2. $a \star (b + c) = (a \star b) \star c$.

Given that $3 \star 2 = 54$, find the value of $5 \star 4$.

Result. 1620

Solution. The second property gives us $5 \star 4 = (5 \star 2) \star 2$. If we denote $f(x) = x \star 2$, then the problem can be reformulated as to find $f(f(5))$, knowing that $f(3) = 54$.

The first property of the star operation immediately gives $f(a + b) = f(a) + f(b)$. So we have $54 = f(3) = f(1) + f(2) = f(1) + f(1) + f(1)$, thus $f(1) = 18$. From this, by induction, we easily obtain $f(n) = 18n$ for every positive integer n. Therefore, $f(5) = 18 \cdot 5$ and $f(f(5)) = 18^2 \cdot 5 = 1620$.

To see that there is actually such an operation, using basic knowledge about exponential functions, we can check that $x \star y = x(3\sqrt{2})^y$ is well-defined for all real numbers x, y and satisfies the given conditions.

Problem 51. Marek painted squares of a grid 10×11 black or white so that each square has at most one neighbouring square (two squares are neighbouring if they share an edge) of the same colour. In how many ways could he do it? Two such coloured grids that look the same only after a rotation are considered different.

Result. 464

Solution. If there is any domino 2×1 of the same coloured squares, then the entire "double" row or "double" column in which this domino lies must also be filled with dominoes in alternating colours. This ensures that all dominoes must be of the same orientation. Recall that filling $n \times 1$ squares with dominoes and squares can be done in $f(n)$ possible ways, where $f(n)$ is the Fibonacci sequence where $f(0) = 1$ and $f(1) = 1$.

Since all dominoes must be of the same orientation and consequent rows or columns are copies of the first row or column, to prevent double counting on the standard chessboard, there must be $f(10) + f(11) - 1$ possible fillings. Furthermore, the colouring of any chessboard can be attained by choosing the colour of the upper left corner and colour the chessboard in an alternating pattern.

Therefore, all possible ways are $2 \cdot (144 + 89 - 1) = 464$.

Problem 52. Helena learned about moving averages. She took her favourite sequence, the Fibonacci sequence $\{F_k\}_{k=0}^{\infty}$, which satisfies $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$, and made a sequence $\{m_k\}_{k=6}^{2024}$ of moving averages, which satisfies $m_k = \frac{F_k + F_{k-1} + \cdots + F_{k-6}}{7}$. How many terms of the sequence $\{m_k\}_{k=6}^{2024}$ are integers?

Result. 252

Solution. Recall the fact that $\sum_{i=0}^{k} F_i = F_{k+2} - 1$. This can be proven by induction: for the first step, $\sum_{i=0}^{0} F_i = F_0 = 0 = 1 - 1 = F_2 - 1$, and for the second step, $\sum_{i=0}^{k+1} F_i = F_{k+1} + \sum_{i=0}^{k} F_i = F_{k+1} + F_{k+2} - 1 = F_{k+3} - 1$. Hence

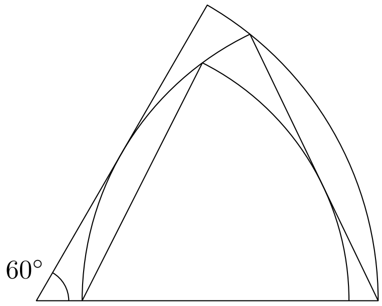
$$F_k + F_{k-1} + \cdots + F_{k-6} = \sum_{i=0}^{k} F_i - \sum_{i=0}^{k-7} F_i = F_{k+2} - F_{k-5} = 7 \cdot m_k.$$

Let d_l be the remainder of F_l after division by 7. The terms of the sequence $\{d_l\}_{l=0}^{2024}$ are

$$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, \ldots 0$$

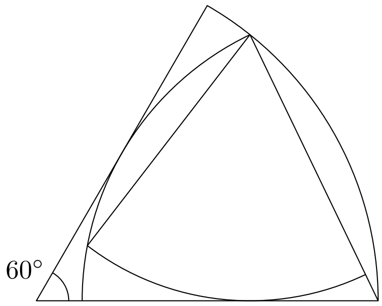
and since $d_l \equiv d_{l-1} + d_{l-2} \pmod 7$, it is clear, that the sequence of remainders d_l has a period of length 16. All indices l such that $d_{l+2} \equiv d_{l-5} \pmod 7$ are $l \equiv 4, 12 \pmod{16}$. Since $6 \leq l \leq 2024$ and $2024 = 126 \cdot 16 + 8$, therefore there are solutions in the form $l = 16 \cdot k + 4$ for $1 \leq k \leq 126$ and also solutions in the form $l = 16 \cdot k + 12$ for $0 \leq k \leq 125$. In total, there are $2 \cdot 126 = 252$ solutions.

Problem 53. Inside a circular sector with a central angle of $60°$, another circular sector is inscribed, and this is done once more, as in the picture. Determine the ratio of the radii of the smallest and largest sectors.



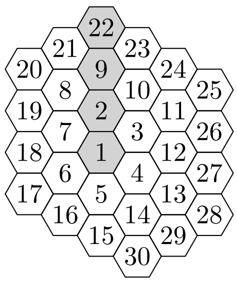
Result. $\sqrt{39}/8$

Solution. Rotate the smallest sector, as shown in the following figure.



From which it is clear that we want to compute the y-coordinate of the intersection of the first and second arcs. Let the centre of the first sector be at coordinates $(0,0)$ and the right-hand vertex at $(1,0)$. Then the largest circle is described via $x^2+y^2=1$ and the medium one via $(x-1)^2+y^2=\left(\frac{\sqrt{3}}{2}\right)^2$. By subtracting the first equation from the second, we obtain $1-2x=\frac{3}{4}-1$, and hence $x=\frac{5}{8}$. Then it can be easily concluded that $y=\pm\frac{\sqrt{39}}{8}$, and since the negative solution does not yield a valid geometric configuration, the only solution is $\frac{\sqrt{39}}{8}$.

Problem 54. There are 2024 hexagonal tiles in the bee hive. In the centre, there is 1 ml of honey. In a spiral pattern, as shown in the figure, there is an increasing volume of honey until the last tile has 2024 ml of honey. Queen Bee decides that she wants to build a highway from the centre tile directly outwards, as shown by the grey colour in the figure. To do that, all the honey from the grey tiles needs to be removed. How much honey (in millilitres) needs to be relocated to build this project?



Result. 17928

Solution. Denote $H(n)$ the amount of honey in the n-th tile from the centre on the desired highway. Then $H(1)=2$, $H(2)=9$ and so on. Take a look at the hexagon formed by tiles at a distance of exactly n from the centre. Walking through any side of this hexagon, we need to do n steps. On the spiral from $H(n)$ to $H(n+1)$, we walk through five sides of the hexagon at a distance of n from the centre and one side of the hexagon at a distance of $n+1$ from the centre. Hence, it can be deduced that $H(n+1)=H(n)+5n+(n+1)=H(n)+6n+1$. Then we can find the close form of this sequence as

$$\begin{aligned}
H(n) &= 6(n-1)+1+H(n-1)=\cdots \\
&= 6\cdot((n-1)+(n-2)+\cdots+1)+(n-1)+H(1) \\
&= 6\cdot\frac{(n-1)n}{2}+n+1 \\
&= 3n^2-2n+1.
\end{aligned}$$

To find the sum of all honey, we need to find the number N of hexagons (excluding the one in the centre) on the highway. Since there are exactly 2024 hexagons, N is the greatest integer satisfying

$$\begin{aligned} H(N) &\leq 2024, \\ 3N^2 - 2N &\leq 2023, \\ N^2 - \frac{2}{3}N &\leq 674 + \frac{1}{3}. \end{aligned}$$

Since $27^2 - \frac{2}{3} \cdot 27 > 729 - 27 > 675$, the value of N can be at most 26 and indeed $26^2 - \frac{2}{3} \cdot 26 < 676 - 18 < 674$, therefore $N = 26$ is the sought-after number of highway hexagons.

Now it remains to calculate the sum

$$\begin{aligned} 1 + \sum_{k=1}^{N} H(k) &= 1 + 3\sum_{k=1}^{N} k^2 - 2\sum_{k=1}^{N} k + \sum_{k=1}^{N} 1 \\ &= 1 + \frac{1}{2}N(N+1)(2N+1) - N(N+1) + N \\ &= 1 + 13 \cdot 27 \cdot 53 - 26 \cdot 27 + 26 \\ &= 17928. \end{aligned}$$

Another way to calculate the final sum is by realising that

$$H(k) = 6 \cdot \frac{(k-1)k}{2} + k + 1 = 6\binom{k}{2} + \binom{k+1}{1}$$

and that recall following identity from Pascal's triangle (known as Hockey-stick identity)

$$\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}.$$

Then,

$$\begin{aligned} 1 + \sum_{k=1}^{N} H(k) &= 1 + 6\sum_{k=1}^{N} \binom{k}{2} + \sum_{k=1}^{N} \binom{k+1}{1} \\ &= 1 + 6\binom{N+1}{3} + \left(\binom{N+2}{2} - 1\right) \\ &= 27 \cdot 26 \cdot 25 + 14 \cdot 27 \\ &= 17928. \end{aligned}$$

Problem 55. How many distinct integers occur in the list

$$\left\lfloor \frac{1^2}{2024} \right\rfloor, \left\lfloor \frac{2^2}{2024} \right\rfloor, \ldots, \left\lfloor \frac{2024^2}{2024} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x?

Result. 1519

Solution. Since $(n+1)^2 - n^2 = 2n+1$, for $n \leq 1011$ it holds that $\frac{(n+1)^2}{2024} - \frac{n^2}{2024} = \frac{2n+1}{2024} \leq \frac{2023}{2024} < 1$ and so $\left\lfloor \frac{(n+1)^2}{2024} \right\rfloor \leq \left\lfloor \frac{n^2}{2024} \right\rfloor + 1$. From that we know the list $\left\lfloor \frac{1^2}{2024} \right\rfloor, \left\lfloor \frac{2^2}{2024} \right\rfloor, \ldots, \left\lfloor \frac{1012^2}{2024} \right\rfloor$ contains all integers from $\left\lfloor \frac{1^2}{2024} \right\rfloor = 0$ to $\left\lfloor \frac{1012^2}{2024} \right\rfloor = 506$, so in the first 1012 terms of sequence, there are 507 distinct elements.

On the other hand, for $n \geq 1012$ it holds that $\frac{(n+1)^2}{2024} - \frac{n^2}{2024} = \frac{2n+1}{2024} \geq \frac{2025}{2024} > 1$ and so $\left\lfloor \frac{(n+1)^2}{2024} \right\rfloor > \left\lfloor \frac{n^2}{2024} \right\rfloor$. Thus, every element from $\left\lfloor \frac{1013^2}{2024} \right\rfloor, \left\lfloor \frac{1014^2}{2024} \right\rfloor, \ldots, \left\lfloor \frac{2024^2}{2024} \right\rfloor$ is new in the list (since it is strictly greater than the previous element), so in the last 1012 terms of sequence, all 1012 of them are distinct (and they are also distinct from the elements in the first half of the sequence).

In total, the sequence contains $507 + 1012 = 1519$ distinct elements.

Problem 56. How many ordered 4-tuples (a, b, c, d) of pairwise distinct numbers $a, b, c, d \in \{1, 2, \ldots, 17\}$ are there such that $a - b + c - d$ is divisible by 17?

Result. 3808

Solution. Construct a regular 17-gon $P_1 \ldots P_{17}$. It follows from $a - b \equiv d - c \pmod{17}$ that P_a, P_b, P_c, P_d form an isosceles trapezium with parallel bases P_aP_c and P_bP_d. After removing one vertex, the remaining 16 vertices can be paired into 8 parallel lines and any two of them can be used as the trapezium (and the corresponding considered subsets $\{a, b, c, d\}$). There are $17 \cdot \binom{8}{2} = 476$ such subsets and each of them defines several ordered 4-tuples: we need to choose which base is P_aP_c and which is P_bP_d and then we can swap a with c and b with d, which gives $2 \cdot 2 \cdot 2 = 8$ options. Therefore, the result is $8 \cdot 476 = 3808$.

Problem 57. Let $ABCD$ be a rectangle and a point E on side CD, such that $2DE = EC$. Let F be an intersection of segments BD and AE. Given that $\angle DFA = 45°$, determine the ratio $\frac{AD}{AB}$.

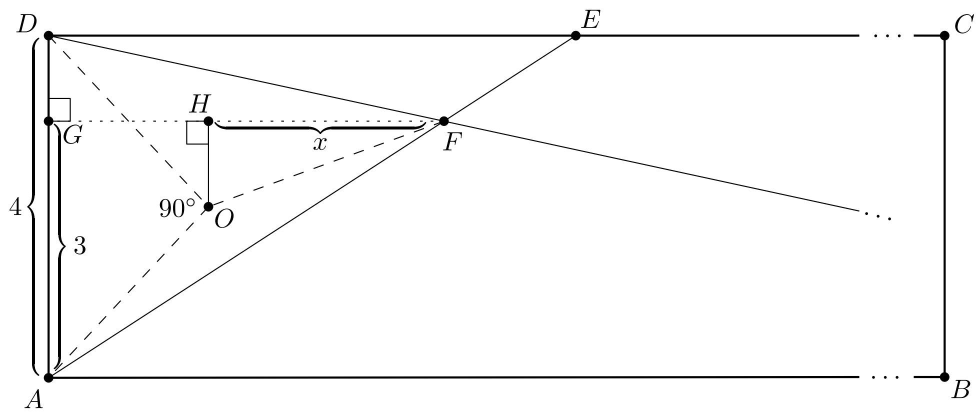
Result. $\frac{\sqrt{7}-2}{3}$

Solution. The configuration is scale invariant, hence we can assume that $AD = 4$. Denote further the perpendicular projection of F on AD by G, the circumcenter of triangle ADF by O and the perpendicular projection of O on GF by H. Triangles ABF and EDF are similar with ratio $AB : ED = 3 : 1$, hence $AG = 3$. We have $\angle DOA = 2 \cdot \angle DFA = 90°$ and hence AOD is right isosceles triangle. The distance of O from both AB and AD therefore equals 2. Denoting the last unknown sidelength in right triangle HOF by $x = HF$ and using Pythagoras' theorem gives

$$x^2 + (3-2)^2 = (2\sqrt{2})^2 = 8 \Rightarrow x = \sqrt{7}.$$

Since triangles DGF and DAB are similar, the sought ratio equals

$$\frac{DA}{AB} = \frac{DG}{GF} = \frac{1}{2+\sqrt{7}} = \frac{\sqrt{7}-2}{3}.$$



Problem 58. Let $P(x)$ be a polynomial of degree 10 with integer coefficients such that it has only real roots and $P(x)$ divides the polynomial $P(P(x) + 2x - 4)$. Find the value of $\frac{P(2024)}{P(206)}$.

Note: Here we say that a polynomial $P(x)$ divides a polynomial $Q(x)$ if $P(x)$ and $Q(x)$ have integer coefficients and there exists a polynomial $R(x)$ with integer coefficients such that $Q(x) = R(x) \cdot P(x)$.

Result. $10^{10} = 10000000000$

Solution. If r is a root of $P(x)$ then all the numbers

$$2r-4, 2(2r-4)-4 = 4r-12, 2(4r-12)-4 = 8r-28, \ldots, 2^n r - 2^{n+2} + 4, \quad n \in \mathbb{N},$$

are also roots of $P(x)$. Since $P(x)$ has at most 10 real roots, there are $j > i$ such that $2^i r - 2^{i+2} + 4 = 2^j r - 2^{j+2} + 4$. This implies $2^i \cdot (r-4) = 2^j \cdot (r-4)$, hence $r = 4$. Thus 4 is the only root and we have $P(x) = a \cdot (x-4)^{10}$, where $a \neq 0$ is a real constant. Hence $\frac{P(2024)}{P(206)} = \left(\frac{2020}{202}\right)^{10} = 10^{10} = 10000000000$.

Problem 59. Joe is standing in the circle of 2024 people labeled clockwise by numbers $1, 2, \ldots, 2024$. They are playing with a frisbee. Person in place 1 throws it to the person in place 3, then they throw it to the person in place 5, and so on. Each person throws the disc to the person next to the one left of them (so they skip one person). The skipped person is now angry that they have not played and leave the circle. This process repeats until the last two people are playing. If Joe wants to be one of those two people in the end, where should he stand in the beginning? Find the sum of numbers of these positions.

Result. 2978

Solution. If the last two people played, then the one holding the disc would toss it to themselves and be the remaining one in the circle. To find the position of this last person, consider following. If there are 2^n persons in the circle, then everyone in the even position will leave after the disc makes one complete round, and we will get a similar situation with 2^{n-1} people. Additionally, the person in position 1 is holding the disc again. Hence, by induction, this one person will be the last. Now if there are $2^n + k$ persons in the circle, then after k tosses, the person holding the disc is in a similar situation with 2^n remaining persons. His position was $2k + 1$. Among $2024 = 1024 + 1000$ people, it will be position 2001. For the second to last one, consider the number of people standing in the circle to be $2^n + 2^{n+1}$, and we claim that the second to last is in position 1. This can be verified for small n: so that the second-to-last remaining person in the circle of 3, 6, or 12 people is person 1. Again, by induction, if there were $2^{n+1} + 2^{n+2}$ people in the circle, then after one circle of tossing, there would remain $2^n + 2^{n+1}$ people, with number 1 holding the disc again. Now for the number of people equal to $2^n + 2^{n+1} + k$ we can deduce that after k tosses, we are in a familiar situation; therefore, the position of the second-to-last person is $2k + 1$ as well. Since $2024 = 1024 + 512 + 488$, we get that the second-to-last position is 977. Hence, the answer is $2001 + 977 = 2978$.

Problem 60. Ondra is throwing frisbee with his three friends. They obey this rule: You cannot pass the disc back to the person who passed the disk to you in the previous turn. Ondra started the game and after ten turns, the disk was held again by Ondra. In how many ways could the ten passes be completed?

Result. 414

Solution. Compute all the possible passing sequences, regardless of Ondra being last. At first, Ondra can pass the disc to three people. Every other friend can only pass the disc to two people because of the rule. If there are n turns, it means there are $3 \cdot 2^{n-1}$ sequences.

Denote the number of sequences that end with Ondra in the n-th turn as y_n. On n-th turn, there are $3 \cdot 2^{n-1}$ sequences in total. In the next turn, some of those sequences can be prolonged to Ondra. Those that cannot be prolonged to Ondra are the ones with Ondra being the one holding disc at n-th turn or $n-1$-th turn because he has to pass disc to someone else. There are y_n and $2y_{n-1}$ of such sequences, respectively; therefore, $y_{n+1} = 3 \cdot 2^{n-1} - y_n - 2y_{n-1}$.

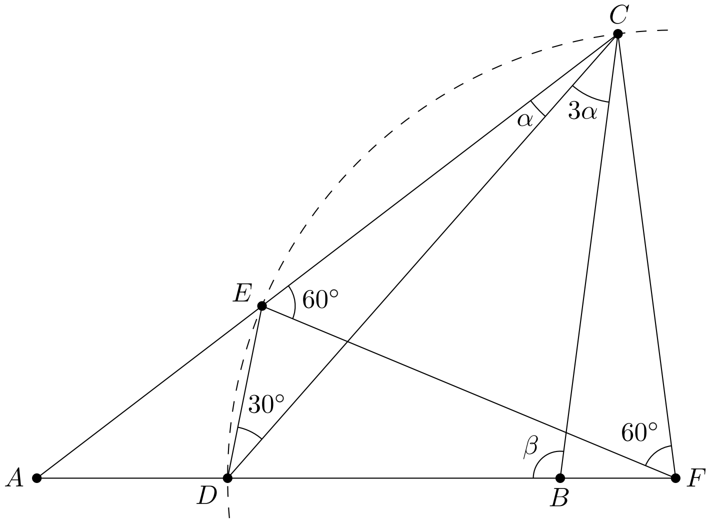
It is more straightforward to calculate each term until y_{10} then search for an explicit formula. From $y_1 = 0$ and $y_2 = 0$, and recurrence relation $y_{n+1} = 3 \cdot 2^{n-1} - y_n - 2y_{n-1}$ it can be calculated that $y_3 = 3 \cdot 2^1 - 0 - 0 = 6$, $y_4 = 3 \cdot 2^2 - 6 = 6$, $y_5 = 3 \cdot 2^3 - 6 - 12 = 6$, $y_6 = 3 \cdot 2^4 - 6 - 12 = 30$, $y_7 = 3 \cdot 2^5 - 30 - 12 = 54$, $y_8 = 3 \cdot 2^6 - 54 - 60 = 78$, $y_9 = 3 \cdot 2^7 - 78 - 108 = 198$, and $y_{10} = 3 \cdot 2^8 - 198 - 156 = 414$.

Alternative solution: Consider a sequence of n passes with Ondra at the beginning, at the end, and nowhere in between. If this sequence happens to be at the beginning of the game, then Ondra can pass to 3 friends, and then it can continue in only 2 ways, after which tosses are deterministic. If such a sequence happens to be in the middle of the game, then one of the friends passes the disc to Ondra; hence, he can choose only from two friends, and then there are only two options left. Such a sequence of n passes cannot be shorter than 3 therefore, it is sufficient to find all partitions of 10 without parts less than 3. There are only the following partitions of 10: 10, $3+7$, $7+3$, $6+4$, $4+6$, $5+5$, $3+3+4$, $3+4+3$, and $4+3+3$. The partition of 10 has 6 possible ways to be played out. Each of the partitions that has two parts can be played out in $6 \cdot 4$ ways; therefore, there are $5 \cdot 6 \cdot 4$ ways. Finally, there are $6 \cdot 4 \cdot 4$ ways how partition with three parts can be played out, hence $3 \cdot 6 \cdot 4 \cdot 4$ ways. Together $6 \cdot (1 + 5 \cdot 4 + 3 \cdot 4 \cdot 4) = 6 \cdot 69 = 414$ ways.

Problem 61. A point D on side AB of triangle ABC is such that $\angle ACD = 11.3°$ and $\angle DCB = 33.9°$. Furthermore, $\angle CBA = 97.4°$. Find $\angle AED$, where E is a point on AC such that $EC = BC$.

Result. $41.3°$

Solution. Put $\alpha = 11.3°$ and $\beta = 97.4°$ for brevity; then $\angle DCB = 3\alpha$. Further, let F be a point on line AB distinct from B such that $CB = CF$.



Let us compute $\angle BCF$: we have $\angle FBC = 180° - \beta$ and $\triangle BCF$ is isosceles, hence $\angle BCF = 180° - 2\angle FBC = 2\beta - 180°$. Now we note that

$$\angle ECF = \alpha + 3\alpha + 2\beta - 180° = 4 \cdot 11.3° + 2 \cdot 97.4° - 180° = 60°.$$

Since $CF = CB$, which equals CE according to the problem statement, $\triangle CEF$ is equilateral.

We further show that $FC = FD$: As $\angle DCF = 60° - \alpha$ and $\angle CFD = 180° - \beta$, we have

$$\angle FDC = 180° - (60° - \alpha) - (180° - \beta) = \alpha + \beta - 60° = 48.7° = 60° - \alpha = \angle DCF,$$

therefore $\triangle CDF$ is isosceles with apex F and $FC = FD$ as desired. Together with $\triangle CEF$ being equilateral this implies $FC = FE = FD$; in other words, points C, E, D lie on a circle with center F. Hence $\angle CDE = \frac{1}{2}\angle CFE = 30°$ and $\angle DEC = 180° - \alpha - 30°$. Finally,

$$\angle AED = 180° - \angle DEC = 30° + \alpha = 41.3°.$$

Problem 62. The real numbers $a > b > 1$ satisfy the inequality

$$(ab+1)^2 + (a+b)^2 \leq 2(a+b)(a^2 - ab + b^2 + 1).$$

Determine the minimum possible value of

$$\frac{\sqrt{a-b}}{b-1}.$$

Result. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Solution. We rearrange the inequality as follows:

$$\begin{aligned}(ab+1)^2 + (a+b)^2 &\leq 2(a+b)(a^2 - ab + b^2 + 1),\\ 0 &\leq 2a^3 + 2b^3 - a^2b^2 - a^2 - b^2 - 4ab + 2a + 2b - 1,\\ 0 &\leq (a^2 - 2b + 1)(2a - b^2 - 1).\end{aligned}$$

Since $a > b > 1$, then also $a^2 > b^2$, and we have $a^2 - 2b + 1 > b^2 - 2b + 1 = (b-1)^2 > 0$. Therefore, for the second bracket, we have

$$\begin{aligned}2a - b^2 - 1 &\geq 0,\\ 2a - 2b &\geq b^2 - 2b + 1,\\ 2(a-b) &\geq (b-1)^2,\\ \frac{\sqrt{a-b}}{b-1} &\geq \frac{1}{\sqrt{2}}.\end{aligned}$$

The value $1/\sqrt{2}$ is obtained e.g. when $a = 5/2$ and $b = 2$ (if we want to get equality, $a = (b^2+1)/2$ has to hold), so it really is the minimal value of $\sqrt{a-b}/(b-1)$.

Problem 63. Let x, y, z be distinct non-zero integers such that

$$\frac{(x-1)^2}{z} + \frac{(y-1)^2}{x} + \frac{(z-1)^2}{y} = \frac{(x-1)^2}{y} + \frac{(y-1)^2}{z} + \frac{(z-1)^2}{x}.$$

Find the minimum possible value of $|64x + 19y + 4z|$.

Result. 7

Solution. Let the symbol $\sum_{\text{cyc}} Q(x,y,z)$ indicate a sum where the other two terms are obtained by repeating twice the cyclic swap $x \to y \to z \to x$ hence, $\sum_{\text{cyc}} Q(x,y,z) = Q(x,y,z) + Q(y,z,x) + Q(z,x,y)$.

Multiplying the equation by $xyz \neq 0$ and rearranging yields

$$P(x,y,z) = x(x-1)^2(y-z) + y(y-1)^2(z-x) + z(z-1)^2(x-y) = \sum_{\text{cyc}} x(x-1)^2(y-z) = 0.$$

Since P vanishes for $x = y$, $y = z$, or $z = x$, it must be divisible by $(x-y)(y-z)(z-x) = \sum_{\text{cyc}} x^2(z-y)$. Since $P(x,y,z)$ is a polynomial of degree 4 and $\sum_{\text{cyc}} x^2(z-y)$ is a polynomial of degree 3, the reminding factor must be linear

$$P(x,y,z) = \left(\sum_{\text{cyc}} x^2(z-y)\right) \cdot (ax + by + cz + d)\,.$$

Furthermore $xy - xz + yz - yx + zx - zy = \sum_{\text{cyc}} x(y-z) = 0$, hence

$$\begin{aligned} P(x,y,z) &= \sum_{\text{cyc}} \left(x^3(y-z) - 2x^2(y-z) + x(y-z)\right) \\ &= \sum_{\text{cyc}} \left(x^3(y-z) - 2x^2(y-z)\right) + 0 \\ &= \left(\sum_{\text{cyc}} x^2(z-y)\right) \cdot (ax + by + cz + d)\,. \end{aligned}$$

From this we can see that a must be -1 so that $x^2(z-y) \cdot ax = x^3(y-z)$ and similarly $b = c = -1$. Further, from $x^2(z-y) \cdot d = -2x^2(y-z)$, we obtain that $d = 2$. Therefore,

$$P(x,y,z) = (x-y)(y-z)(z-x)(2-x-y-z) = 0.$$

As we only look for pairwise distinct triples (x,y,z), it must follow that $x+y+z = 2$. It is easy to see that any triple with these properties solves the original equation as well.

To find the minimal value of the expression $|64x + 19y + 4z|$, subtract $4(x+y+z) - 8 = 0$ to get

$$|64x + 19y + 4z| = |15 \cdot (4x + y) + 8|.$$

We are searching for an integer $4x + y$ that minimises the expression. The minimum is clearly attained for $4x + y = -1$, for instance, at $(x,y,z) = (-2, 7, -3)$. Therefore, the result is 7.