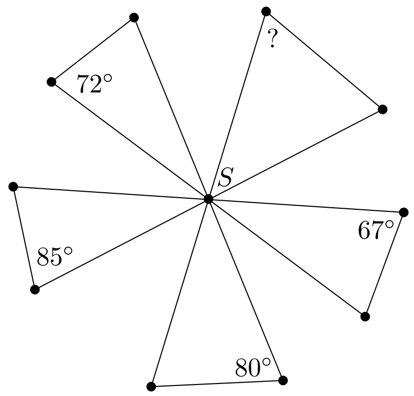
Problem 1. When asked about her age, grandmother Mary gives the answer in a puzzle: I have five children and they are of different age, each 4 years apart. I had my first child when I was 21 years old, and now my youngest child is 21. Find the age of grandma.

Result. 58

Solution. Clearly, the ages of the children are $21, 21+4, \ldots, 21+4\cdot 4$. Grandma is 21 years older than her oldest offspring, namely she is $21+4\cdot 4+21 = 58$ years old.

Problem 2. An old windmill propeller consisting of five triangular blades is formed by five solid line segments of equal length, whose midpoints all lie at point S and whose endpoints are connected as in the picture. Determine the size of the angle denoted by the question mark in degrees.



Result. 56

Solution. The ten angles adjacent to point S form five pairs of opposite angles of equal size and the five of them which are the interior angles at the vertex S in the five isosceles triangles sum up to $180°$. Hence the sum of the five marked angles equals $\frac{5\cdot 180° - 180°}{2} = 360°$ and it follows that the size of the missing angle is $360° - 67° - 80° - 85° - 72° = 56°$.

Problem 3. Students have an opportunity to take part in three different athletics competitions. Each student has to take part in at least one competition. In the end, 22 students have chosen the sprint race, 13 students have gone for the long jump and 15 students have taken part in the shot put competition. Furthermore, we know that 8 students have selected the sprint race and the long jump, 7 students have chosen the sprint race and the shot put, and 6 students have decided in favor of long jump and shot put. There are 3 very ambitious students who have taken part in all three competitions. How many students are there in this class?

Result. 32

Solution. Add the number of participants of the single competitions and subtract the number of students who have chosen two competitions from the result. Hence, the very motivated students who have taken part in all three competitions are subtracted once too often, their number has to be added. Therefore, the final result is $22+13+15-8-7-6+3 = 32$.

Problem 4. A number is called *super-even* if all of its digits are even. How many five-digit super-even numbers are there such that when added to 24680 the result is also super-even?

Result. 90

Solution. There are five even one-digit numbers. To guarantee that the result is super-even, any two digits added during the traditional addition algorithm must lead to a sum less than 10. Bearing in mind that a number cannot start with zero, we have three possibilities (2, 4 and 6) for the first digit, another three possibilities for the second digit, two possibilities for the third digit, one possibility at the fourth digit and five possibilities for the last one. Since these choices are independent, we get $3\cdot 3\cdot 2\cdot 1\cdot 5 = 90$ such numbers in total.

Problem 5. Once upon a time there was a wise King. His castle was in the centre of four concentric circular walls of radii 50, 100, 150, 200 and the land surrounded by the largest wall was used as the castle grounds (including the land inside the other walls). There were peaceful times, so he decided to tear down all four walls and build only one circular wall, again with his castle as the center, of maximal possible radius from the material of the old ones. What is the ratio of the area of the new castle grounds and the area of the old one (as a number greater or equal to one)?

Result. $\frac{25}{4}$

Solution. We know that the sum of the perimeters of the four circular walls will be the perimeter of new circular wall. Denote the radius of the new circular wall by r. Then

$$2\pi \cdot 50 + 2\pi \cdot 100 + 2\pi \cdot 150 + 2\pi \cdot 200 = 2\pi \cdot r$$

implies that r is the sum of the given radii, i.e. $r = 500$. Hence, the desired ratio is $\frac{\pi \cdot 500^2}{\pi \cdot 200^2} = \frac{25}{4}$.

Problem 6. Zoe is trying to open a lock. She knows the following about its four-digit code:

- all of its digits are different,
- 137 and 17 divide it,
- the sum of its digits is the smallest possible prime.

What is the code?

Result. 9316

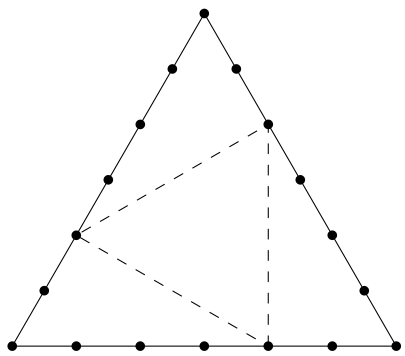
Solution. Since the sought number is divisible by the primes 137 and 17, it must be a multiple of $137 \cdot 17 = 2329$. Observe that this number does not comply with the conditions and that this number only can be multiplied by 2, 3, or 4 to stay a four-digit number. We compute $2329 \cdot 2 = 4658$, $2329 \cdot 3 = 6987$ and $2329 \cdot 4 = 9316$, which have sums of digits 23, 30, and 19, respectively. Because 19 is the smallest occurring prime, the number 9316 opens the lock.

Problem 7. There are four polygons on the table – an equilateral triangle with unit side length and three other congruent regular polygons with unit side lengths as well. Every two of the four polygons share exactly one side and no two of them overlap. What is the perimeter of the resulting shape, not counting the shared sides?

Result. 27

Solution. Let us assume the congruent polygons have n sides each. Then the resulting shape has $3(n-3)$ sides, since three sides of all polygons including the triangle are shared. We only need to determine n. Since the outer angle of the equilateral triangle is $300°$, the inner angle of the congruent polygons must be $150°$. Since the sum of inner angles of any $n-$gon equals $(n-2) \cdot 180°$, we have to solve the equation $150n = 180(n-2)$, which holds for $n = 12$. Plugging into the above formula, we obtain that the resulting polygon has $3 \cdot 9 = 27$ sides.

Problem 8. There is an equilateral triangle with several marked points (including its three vertices) on its boundary dividing each of its sides into 2021 congruent segments. Determine the number of all equilateral triangles with vertices in these marked points. The figure shows one of such triangles in case that each side of the given equilateral triangle was divided into 6 congruent parts.



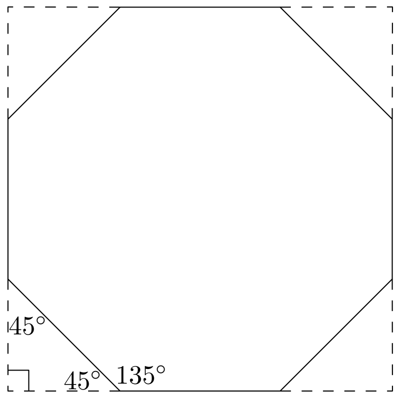
Result. 8081

Solution. Let us refer to equilateral triangles just as to triangles. There is the original triangle, then $3 \cdot 2020$ triangles sharing exactly one vertex with the original triangle (2020 for each vertex) and there are 2020 rotated triangles (as on the picture from the statement) sharing no vertex with the original one. It is easy to see that these triangles are all distinct and there is no other such triangle. Altogether, we have $1 + 3 \cdot 2020 + 2020 = 8081$ desired triangles.

Problem 9. Veronica cuts off the four corners of a square sheet of paper in such a way that a regular octagon remains. The cut off material has a total area 300. What is the side length of the regular octagon?

Result. $\sqrt{300} \approx 17.32051$

Solution. The interior angles of regular octagon are 135°. Therefore Veronica cuts off four right-angled isosceles triangles which may be assembled in a square with the side length of the regular octagon. As a consequence, the side length of the regular octagon equals $\sqrt{300}$.



Problem 10. Find the largest three-digit positive integer n with the following properties:

- the sum of digits of n is 16,
- the product of digits of n is not 0, but the units digit of this product is 0,
- the sum of digits of the product of digits of n is 3.

Result. 853

Solution. From the second condition we obtain that at least one digit of n has to be 5 and at least one digit has to be even. However, none of the digits may be 0. Taking this into consideration, we derive the possibilities 5, 2, 9 or 5, 4, 7 or 5, 6, 5 or 5, 8, 3 from the first condition. From these values only 5, 8, 3 fulfill the last condition and hence the largest three-digit integer satisfying the conditions is 853.

Problem 11. Exactly five digits are to be removed from the number 6437051928 so that the resulting five-digit number is the largest possible. What will the resulting number be?

Result. 75928

Solution. The largest ten-thousand-digit that can be achieved by removing at most five digits from the left is 7, so we need to remove the first three digits. By analogous arguments we conclude that the optimal choice is then finished by removing digits 0 and 1. Therefore, 75928 is the sought answer.

Problem 12. Let n be a positive integer. Now consider the increasing sequence S_n starting with 1 and having constant difference n between one term and the next term. For instance, S_2 is the sequence $1, 3, 5, \ldots$ For how many values of n does S_n contain the number 2021?

Result. 12

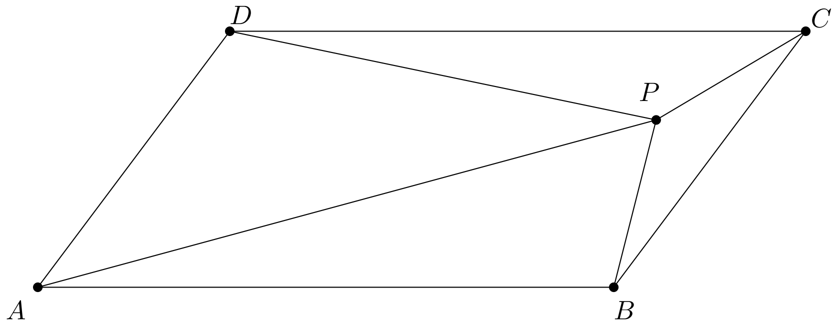
Solution. The number 2021 appears as a term in the sequence S_n if and only if $2021 = 1 + an$ for some positive integer a. In other words, $2020 = an$, so n has to be a divisor of 2020. The prime factorization of 2020 is $2020 = 2^2 \cdot 5 \cdot 101$. Any divisor of 2020 is obtained by multiplying some of these primes. We can take the prime 2 zero, one or two times, that gives 3 possibilities. The prime 5 can be taken or not – giving 2 possibilities and the same holds for 101. In total, we have $3 \cdot 2 \cdot 2 = 12$ possibilities how to choose a divisor of 2020. Hence 2021 is a term in 12 sequences S_n.

Problem 13. There is a straight 90-meter-long corridor with ten windows, each two neighbouring being 10 meters apart. Tommy placed seven robots by seven different windows, and switched them all on at the same time. When switched on, each robot moves at a constant speed of 10 meters in a minute in one of the two directions until it reaches the end of the corridor where it instantly turns and continues in the same manner in the opposite direction. Tommy was measuring the time until the first moment when every robot met all the others. Determine the largest possible value he could have measured in seconds.

Result. 510

Solution. For a given robot A we can determine the window and direction of a robot which would meet A in the longest possible time – it is the window right behind A heading in the opposite direction (if placed by one of the end windows, we assume that a robot is heading out of the corridor in this argument). The time to the first meeting is then easily computed to be 8.5 minutes which is equal to 510 seconds.

Problem 14. Inside the parallelogram $ABCD$ there is a point P such that the area of the triangle CDP is three times the area of the triangle BCP and one third the area of the triangle APD. Find the area of the triangle ABP if the area of the triangle CDP is 18.



Result. 42

Solution. From the formula for the area of a triangle "side length times the corresponding altitude divided by two" we deduce that triangles APD and BCP cover half of the area of the parallelogram. Therefore the area of triangle ABP equals

$$\left(\frac{1}{3}+3\right)\cdot 18-18=42.$$

Problem 15. Dividing 1058, 1486 and 2021 by a certain positive integer $d>1$ leaves always the same remainder. Find the number d.

Result. 107

Solution. The distances between the three numbers are $1486-1058=428$ and $2021-1486=535$. Since the numbers given leave the same remainder when divided by d, the distances have to be multiples of d. The greatest common divisor of 428 and 535 is the prime 107, which is the sought integer d.

Problem 16. In the football stadium the substitutes' bench has fourteen single chairs. The new management of the team consisting of coach, assistant coach, manager and physiotherapist wants to become acquainted with all the players. Therefore, during the game they want to sit on the bench among the ten substitute players in such a way that each member of the management sits between two players. In how many ways can the members of the management choose their four chairs to achieve this?

Note: Using two different orders on the same four chairs counts as two different ways.

Result. 3024

Solution. Imagine the ten substitute players standing in a row. There are nine gaps between the ten players and each gap can be occupied by at most one member of the trainer team. Therefore they have $9\cdot 8\cdot 7\cdot 6=3024$ possibilities to sit in the desired way.

Problem 17. A regular pyramid has a square base of area 1. The surface area of the whole pyramid equals 3. What is its volume?

Result. $\frac{\sqrt{3}}{6}\approx 0.288675$

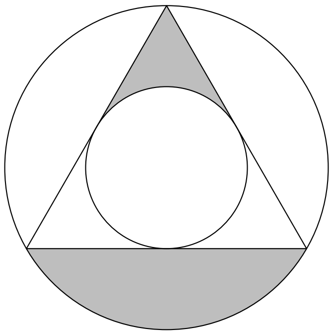
Solution. The base has sides of size 1. As the whole pyramid has surface area 3, each of its four triangular faces has the area $\frac{1}{2}$ and height equal to 1. Thus a cut through its vertex and heights of two opposite faces creates an equilateral triangle of side 1 whose height $\frac{1}{2}\sqrt{3}$ is the same as the height of the pyramid. The volume of the pyramid equals one third of the base surface area times the height, namely $\frac{1}{3}\cdot 1\cdot\frac{1}{2}\sqrt{3}=\frac{1}{6}\sqrt{3}$.

Problem 18. The magical Cana machine transforms liquids. If it gets pure water, it converts 6 % of it to wine and keeps the remaining 94 % unchanged. If it gets pure wine, it converts 10 % of it to water and keeps the remaining 90 % unchanged. If it gets a mixture, it acts on the components separately as described above. Mary bought water and wine, 6000 liters in total, and poured everything into her Cana machine. After the machine had stopped, Mary realised that the mixture remained unchanged. How many litres of wine were there?

Result. 2250

Solution. Denote by x the amount of wine, and by z the amount of water in litres before the Cana machine was used. We know that $0.06z$ litres of water gets converted into wine and $0.1x$ litres of wine to water and that x is the same before and after running the machine. Hence, we must have $0.06z=0.1x$. Given that $x+z=6000$, we can write $0.06(6000-x)=0.1x$. Solving for x leads to $6000\cdot\frac{3}{8}=2250$ litres of wine.

Problem 19. The figure shows an equilateral triangle with its incircle and its circumcircle. Find the area of the shaded region, if the area of the circumcircle is 140.



Result. 35

Solution. It is easy to compute that the radius of the incircle of an equilateral triangle is half the radius of its circumcircle. Thus the area of the incircle equals $\frac{140}{4} = 35$. Furthermore, the shaded area is exactly one third of the annulus given by the two circles, i.e. 35 as well.

Problem 20. If the product of 2021 positive integers equals twice their sum, what is the largest possible value of one of them?

Result. 4044

Solution. Let us denote the positive integers as $c_1 \geq c_2 \geq \cdots \geq c_{2020} \geq c_{2021} \geq 1$. We wish to determine the largest possible value of c_1 assuming that the equation

$$c_1 \cdots c_{2021} = 2 \cdot (c_1 + \cdots + c_{2021}) \tag{1}$$

holds. Dividing by the left-hand side and estimating the denominators from below by setting some of the numbers c_i to 1 yields

$$1 = 2\left(\frac{1}{c_2 \cdots c_{2021}} + \cdots + \frac{1}{c_1 \cdots c_{2020}}\right) \leq 2\left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{2019}{c_1 c_2}\right) = 2 \cdot \frac{2019 + c_1 + c_2}{c_1 c_2}.$$

Multiplying by $c_1 c_2$ and rearranging then gives

$$(c_1 - 2)(c_2 - 2) = c_1 c_2 - 2c_1 - 2c_2 + 4 \leq 2 \cdot 2019 + 4 = 4042.$$

If $c_2 \geq 3$ we get $c_1 \leq 4044$ and the choice $c_1 = 4044$, $c_2 = 3$ and $c_3 = \cdots = c_{2021} = 1$ satisfying the equation (1) shows that this value can be attained. On the other hand, if $c_2 \leq 2$ then the numbers $c_2 \geq c_3 \geq \cdots \geq c_{2021}$ consist of $k \geq 0$ twos and $2020 - k$ ones and the equation (1) reads as

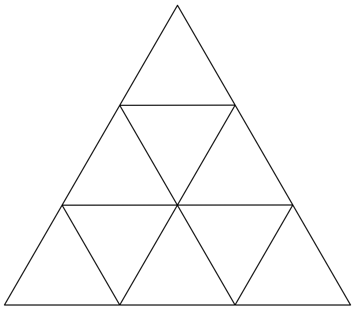
$$c_1 2^k = 2(c_1 + 2k + 2020 - k).$$

It can be further simplified to $c_1(2^{k-1} - 1) = 2020 + k$ and we observe that for $k \leq 1$ there is no c_1 satisfying this equation and for $k \geq 2$ we have

$$c_1 = \frac{2020 + k}{2^{k-1} - 1} \leq 2022 < 4044,$$

so the result is 4044.

Problem 21. The nine small triangles in the picture below are to be filled-in with distinct positive integers such that any two numbers in triangles sharing a side have a common divisor greater than 1. What is the smallest possible sum of the nine numbers filled in?



Result. 59

Solution. First of all, observe that there are three cells with one neighbour, three ones with two and three ones with three neighbours. This means, if a prime p is one of the nine numbers, at least one multiple of p has also to be among the nine numbers. Secondly, note that 1 can not be one of the nine numbers. Denote the sum of a valid filling by S.

If there is a prime $p \geq 11$ in a valid filling, then at least one multiple $k \cdot p$ with $k \geq 2$ must also be present. Fill the remaining seven cells by the smallest available numbers, regardless whether they comply with the rules. Then

$$S \geq 2 + 3 + 4 + 5 + 6 + 7 + 8 + p + k \cdot p = 35 + (k+1) \cdot p \geq 35 + 33 = 68.$$

Now we assume no prime $p \geq 11$ is present in a valid filling and consider four subcases:

- Both the numbers 5 and 7 are present:

$$S \geq 5 + k_5 \cdot 5 + 7 + k_7 \cdot 7 + 2 + 3 + 4 + 6 + 8 = (k_5 + 1) \cdot 5 + (k_7 + 1) \cdot 7 + 23 \geq 15 + 21 + 23 = 59.$$

- Number 5 is present, but there is no 7:

$$S \geq 5 + k \cdot 5 + 2 + 3 + 4 + 6 + 8 + 9 + \begin{cases} 10 \geq 20 + 32 + 10 = 62 & \text{for} \quad k \geq 3, \\ 12 = 15 + 32 + 12 = 59 & \text{for} \quad k = 2. \end{cases}$$

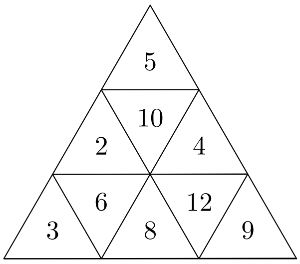
- There is neither 5 nor 7:

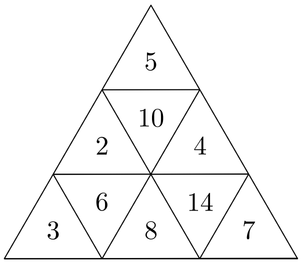
$$S \geq 2 + 3 + 4 + 6 + 8 + 9 + 10 + 12 + 14 = 68.$$

- Number 7 is present, but there is no 5:

$$S \geq 2 + 3 + 4 + 6 + 7 + 8 + 9 + 10 + k \cdot 7 \geq 49 + 14 = 63.$$

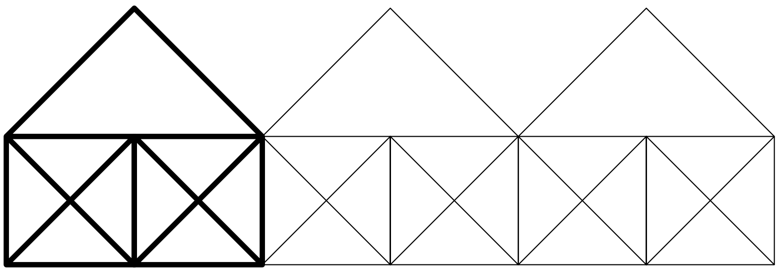
Altogether, we can conclude that 59 is a candidate for the minimal sum. In fact, both sets of numbers from above having sum 59 can be filled in according to the rules:





Therefore, the answer is 59.

Problem 22. Lotta is repeatedly drawing the same house: it consists of two congruent squares and an isosceles right-angled triangle serves as the roof. Each new house is put in line next to the existing ones. Here you can see her first three houses:



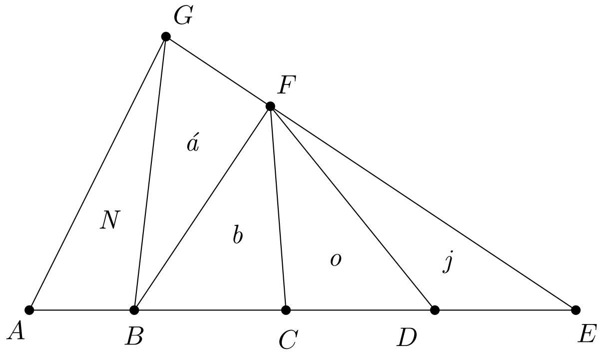
What is the minimum number of houses she has to draw in order to count at least 2021 triangles in her drawing?

Result. 93

Solution. Define the area of one house to be 3. In the first house, there are 8 triangles of area $\frac{1}{4}$, 8 triangles of area $\frac{1}{2}$, and 3 triangles of area 1. These add up to 19. In the second house, we can find the same number of triangles as in a single house plus 2 triangles of area 1 which reach from one house into the other. Therefore, the second house provides 21 triangles.

Starting at house number 3, each additional house contributes the number of triangles of the second house plus one triangle of area 4 which ranges over three houses. So, for each additional house, there are 22 triangles. Since $2021 - 19 - 21 = 1981$ and $1981 = 90 \cdot 22 + 1$, Lotta has to draw $2 + 90 + 1 = 93$ houses.

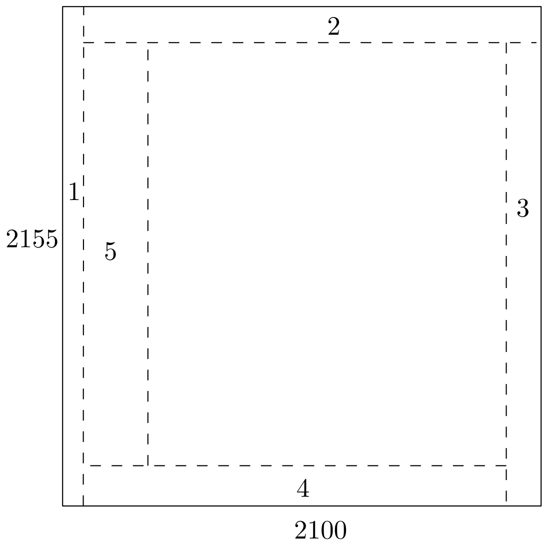
Problem 23. Each of the five triangles N, $\acute{a}$, b, o, j has the same area. Find AB if $CD = 5$.



Result. $\frac{15}{4}$

Solution. The area ratio between the triangles BEG and BEF is $4:3$. Since these triangles have the same base line BE, the respective heights must have the ratio $4:3$, too. Since the triangles ABG and CDF have the same area, we conclude that $AB = \frac{3}{4}CD = \frac{15}{4}$.

Problem 24. Anna has a large rectangular sheet of paper with side lengths 2155 and 2100. She cuts off a strip of width 1 along the longer side, then, continuing clockwise, a strip of width 2 along the shorter side and again a strip of width 3 along the longer side. She continues to cut off strips of widths increasing by one as long as this is possible, see the following picture.



Eventually, she ends up with a rectangle from which she cannot cut off any strip of increasing width anymore. Find the area of this rectangle.

Result. 6375

Solution. Anna may cut off strips of odd width as long as

$$1 + 3 + \cdots + 2n - 1 = n^2 < 2100.$$

Since

$$45^2 = 2025 < 2100 < 2116 = 46^2,$$

the strip of width 89 is the last one possible having odd width. Furthermore, she may cut off strips of even width as long as

$$2 + 4 + \cdots + 2n = n(n+1) < 2155.$$

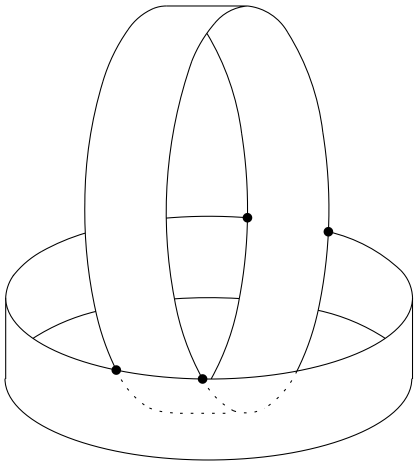
Due to

$$45 \cdot 46 = 2070 < 2155 < 2162 = 46 \cdot 47,$$

the strip of width 90 is the last one possible having even width. Therefore, as she can cut off all 90 strips, the remaining rectangle has the area

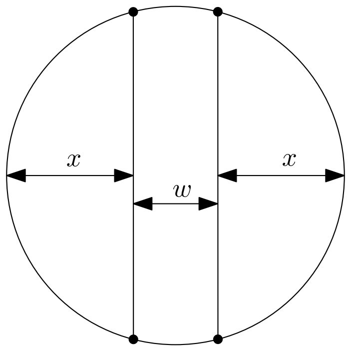
$$(2100 - 2025) \cdot (2155 - 2070) = 75 \cdot 85 = 6375.$$

Problem 25. One of two identical rings of radius 4 and unknown width w lies horizontally on a table while the second one is oriented vertically, it touches the first one at exactly four points (see the figure) and its lowest point lies at the height 1 above the table. What is w?



Result. $\frac{10}{3}$

Solution. Let us denote the desired width by w and look at the figure depicting the vertical projection of the two rings onto the table.



The altitude of the lowest point(s) of the vertical ring above the table is then $1 = w - x$. Hence

$$8 = 2x + w = 2(w - 1) + w = 3w - 2$$

and thus $w = \frac{10}{3}$.

Problem 26. A polynomial of degree 14 has integer coefficients, the leading one being positive, and 14 distinct integer roots. Its value p at zero is positive. Determine the lowest possible p.

Result. 29030400

Solution. The polynomial can be written as $c \cdot (x - a_1) \cdot (x - a_2) \cdots (x - a_{14})$ for some pairwise distinct integers $a_1, a_2, \ldots, a_{14}$ and a number c. The leading coefficient is then equal to c, so c is positive. The value at 0 equals the constant coefficient, that is the product $c \cdot a_1 \cdot a_2 \cdots a_{14}$. As we want to minimize it, we set c as small as possible, that is $c = 1$. To minimize the remaining product of the roots, we have to take them as close to zero as possible, that is $1, -1, 2, -2, \ldots$ However, we have to choose an even number of negative integers, which leads to the result $6! \cdot 8! = 29030400$.

Problem 27. Beata writes the digits 4, 5 and 7 using two strokes and every other digit using one stroke. How many strokes does she use when she writes all integers from 1 to 2021 including these two numbers?

Result. 8783

Solution. When she writes the integers from 1 to 2021, she writes 9 one-digit numbers, 90 two-digit numbers, 900 three-digit numbers and $2021 - (9 + 90 + 900) = 1022$ four-digit numbers. In total, she writes

$$9 + 90 \cdot 2 + 900 \cdot 3 + 1022 \cdot 4 = 6977$$

digits. For each digit, she does one stroke, while she does one additional stroke for each digit which equals 4, 5 or 7. So it is sufficient to count how many of these digits are written. Since the number 2021 contains none of the digits 4, 5 or 7, we consider only integers up to 2020.

Let us count the digits 4 she writes. There is $\frac{1}{10}$ of all numbers up to 2020 which has the digit 4 in the ones place, i.e. 202 numbers. The digit 4 occurs in the tens place in $\frac{1}{10}$ of the numbers up to 2000 and in no number between 2001 and 2020. Altogether that is 200 times. Similarly, the digit 4 occurs in hundreds place 200 times. In total, the digit 4 is written $202 + 200 + 200 = 602$ times. The same goes for the digits 5 and 7.

To conclude, she writes 6977 digits while $3 \cdot 602 = 1806$ of them are 4, 5 or 7. Therefore, she does

$$6977 + 1806 = 8783$$

strokes.

Problem 28. The table below should be filled with the numbers $1, 1, 2, 2, \ldots, 8, 8$ in such a way that for every used number n there are exactly n other cells between the two occurrences of n. Three of these numbers are already placed in advance:

					6	7		2							

Insert the remaining numbers according to the rules and give the 4-digit integer number in the shaded area as a solution. For $1, 1, 2, 2, 3, 3$ a correctly filled example would be:

3	1	2	1	3	2

Result. 3845

Solution. For the ease of notation we treat the table given as an array f with sixteen entries. From the three entries given, $f(6) = 6$, $f(7) = 7$, $f(9) = 2$, we uniquely get $f(13) = 6$, $f(15) = 7$, and $f(12) = 2$.

					6	7		2			2	6		7	

In principle there are two different strategies: either to look, where a specific pair of numbers may be placed, or to consider which numbers are possible in a specific cell like in Sudoku.

For example we can look for the possibilities to fill in the pair of 3. There are three possibilities: either $f(1) = f(5) = 3$ or $f(4) = f(8) = 3$ or $f(10) = f(14) = 3$.

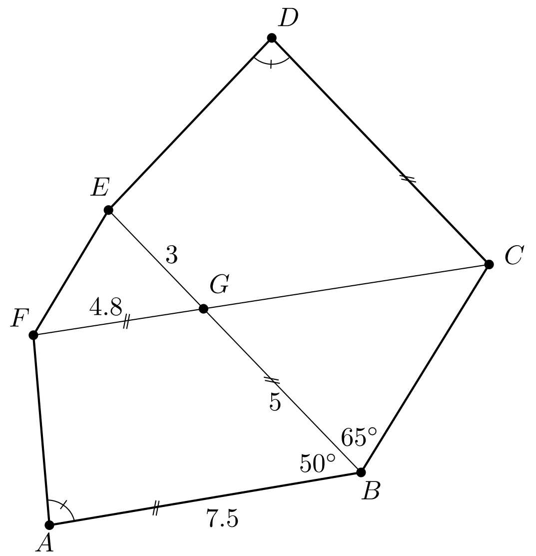
It is easy to see that $f(10) = f(14) = 3$ leaves the only possibility $f(16) = 4$ to fill $f(16)$ and, as a consequence, we get $f(11) = 4$. But now we cannot place the pair of 8 anymore. The case $f(4) = f(8) = 3$ results in two possibilities for the pair of 5, namely $f(5) = f(11) = 5$ or $f(10) = f(16) = 5$. Both alternatives produce a contradiction immediately, as we can't place the pair of 4 in neither case.

Now $f(1) = f(5) = 3$ leaves the unique possibility $f(2) = f(11) = 8$ and the only way to fill the remaining cells according to the rules is $f(3) = f(8) = 4$, $f(14) = f(16) = 1$ and finally $f(4) = f(10) = 5$.

3	8	4	5	3	6	7	4	2	5	8	2	6	1	7	1

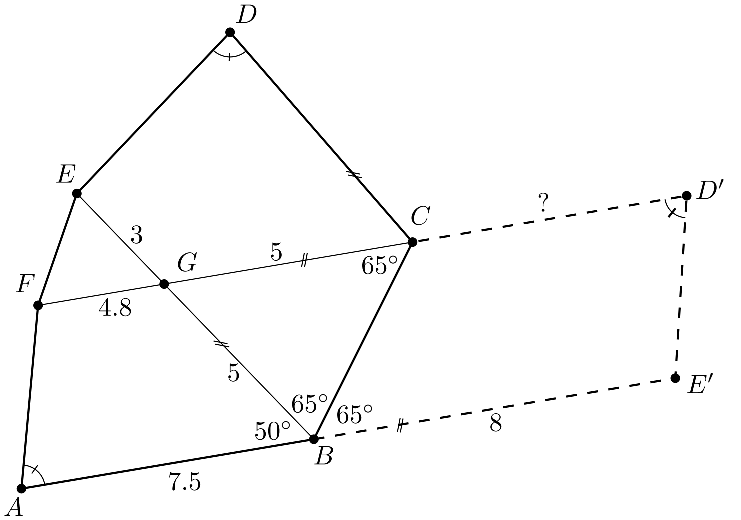
Therefore the unique solution asked for is 3845.

Problem 29. A convex hexagon $ABCDEF$ with intersection G of its diagonals BE and CF satisfies the following properties as indicated in the sketch: $AB = 7.5$, $BG = 5$, $GE = 3$, $GF = 4.8$, $\angle BAF = \angle CDE$, $\angle ABG = 50°$, $\angle CBG = 65°$, AB is parallel to CF, and CD is parallel to BE. Determine the length of CD.



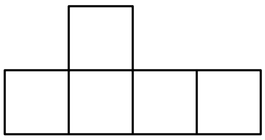
Result. $5.7 = \frac{57}{10}$

Solution.



Since $50° + 65° + 65° = 180°$ and due to the two pairs of parallel segments, this is exactly the configuration obtained by folding the parallelogram $AE'D'F$ along a suitable segment BC (it is indeed a parallelogram as $\angle BAF = \angle CDE$; see the sketch). Equivalently, one gets the same result by mapping the vertices D, E along the line BC. Observing moreover that the triangle BCG is isosceles, the length conditions yield $CD = AB + BE' - CF = AB + EG - FG = 5.7$.

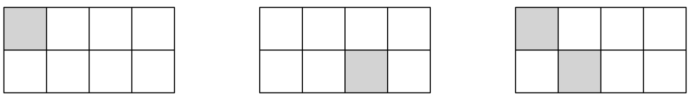
Problem 30. Nadja and Selina are playing the game *Battleship*. Among other ships, everybody has got a battle cruiser with an airfield for helicopters of the following shape:



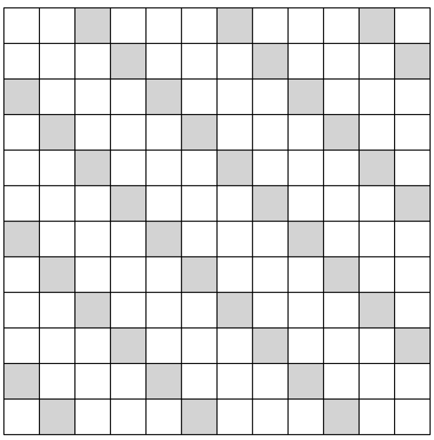
Nadja hides her battle cruiser somewhere in the 12×12 grid of the playing field. Since they are playing the game using pencil and paper, the above figure of the battle ship can be rotated and flipped over. At least how many times does Selina have to shoot, i.e. to select a square of the grid, to be certain to hit Nadja's battle ship at least once?

Result. 36

Solution. Consider a 4×2 block. As can be seen in the first two pictures, choosing exactly one square does not guarantee a hit, since there is still an option to place the battle cruiser without being hit. Furthermore, it is clear that the choice of any square in one row and another square in the other row prevents hiding the battle cruiser in this block. The third picture shows an example for that. Therefore, two shots into a 4×2 block are necessary, yielding at least $18 \cdot 2 = 36$ shots in total.



On the other hand, the following diagonal pattern of the 12×12 grid demonstrates that 36 shots are also sufficient to ensure at least one hit on the battle cruiser.



Problem 31. For a fixed positive integer a we construct an acute-angled triangle ABC such that $BC = a$ and lengths h_b, h_c of the corresponding altitudes are integers as well. Given that the greatest possible area of such triangle is 101.4, determine a.

Result. 13

Solution. As ABC is acute-angled, we obtain $h_b < a$ and $h_c < a$. On the other hand, it is easy to see that in order to maximize the area $S = \frac{1}{2}ah_a$, both altitudes h_b and h_c should be as long as possible (indeed, prolongation of, say, the altitude h_b keeping the condition $h_b < a$ with h_c fixed makes h_a longer as well). Therefore $h_b = h_c = a - 1$ and ABC is isosceles. Let us denote the midpoint of BC by M and the foot of the altitude h_c by C_0. From the Pythagorean theorem for similar right triangles $ABM \sim CBC_0$ we compute

$$101.4 = S = \frac{1}{2}ah_a = \frac{a^2(a-1)}{4\sqrt{2a-1}}.$$

Since 101.4 is a rational number, the integer $2a - 1$ must be an odd square, say $(2k + 1)^2$ for some integer $k \geq 0$, and hence $a = \frac{(2k+1)^2+1}{2} \in \{1, 5, 13, 25, 41, \dots\}$. Checking the first few values and realizing that the area is strictly increasing in a, we quickly find the correct answer $a = 13$.

Problem 32. Ludwig computed the sum of 1000 positive integer numbers and got the value 1 200 500. If these numbers are arranged in ascending order, then the difference between two consecutive numbers is either 2 or 7. The smallest of the numbers is 101. Now he wants to maximize the largest number within this sum by keeping all other conditions fulfilled. What is the maximum value possible for this number?

Result. 3099

Solution. Let $101 = n_1, n_2, \dots, n_{1000}$ denote the positive integers. Assuming that the difference between two consecutive numbers would always be 2, we get

$$n_{1000} = 101 + 2 \cdot 999 = 2099,$$
$$\sum_{i=1}^{1000} n_i = 2200 \cdot 500 = 1\,100\,000.$$

Therefore, the difference between Ludwig's sum and the minimal possible sum is $100\,500 = 20\,100 \cdot 5$. If $n_{i+1} - n_i = 7$ instead of 2, then the sum is increased by $(1000 - i) \cdot 5$ and n_{1000} is increased by 5. As a consequence, in order to maximize n_{1000} considering a constant sum of 1 200 500, the distances of 7 should be put at the highest indices possible. Luckily, the value 20 100 is the triangular number $\frac{1}{2} \cdot 200 \cdot 201$. Therefore, if Ludwig changes the distances $n_{i+1} - n_i$ for $i = 800, \dots, 999$ to 7 instead of 2 compared to the minimal sum above, he gets the largest possible

$$n_{1000} = 101 + 2 \cdot 999 + 200 \cdot 5 = 3099.$$

Problem 33. What is the smallest positive integer that can be written using only the digits 2 and 9, has an odd number of digits, and is divisible by 11?

Result. 29 292 929 292

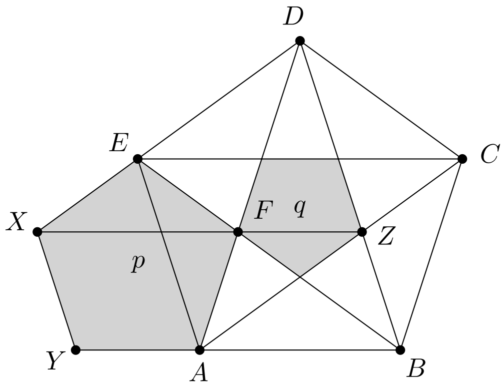
Solution. Our solution requires a little modular arithmetic. Observe that $100 \mod 11 = 1$. This means that multiplying by 100 does not change a number's remainder modulo 11. Now suppose a number's decimal expansion contains the same digit a twice in a row. This number has the form $x \cdot 10^{n+2} + 11a \cdot 10^n + y$. Crossing out the consecutive a's we get the number $x \cdot 10^n + y$ which has the same remainder modulo 11. All of this means that our answer cannot contain a pair of consecutive 2's or 9's, for otherwise we could cross them out and get a smaller positive integer, still with an odd number of digits, and still divisible by 11. Now we simply try longer and longer strings of the form 2929 . . . 292 or 9292 . . . 929 until we find the smallest such number that is divisible by 11.

The solution is even easier to find if one knows the divisibility criterion for 11. A number is divisible by 11 if and only if the sum of the digits in even positions minus the sum of the digits in odd positions is divisible by 11. This immediately implies that our answer cannot contain a pair of consecutive 2's or 9's, for otherwise we could cross them out and get a smaller solution, as before. Since 2's and 9's must alternate, we look for the smallest n such that $2n - 9(n + 1)$ is a multiple of 11, or $9n - 2(n + 1)$ is. In each case the solution is $n = 5$, which yields the candidates 29 292 929 292 and 92 929 292 929. We take the smaller of these two numbers.

Problem 34. Let $ABCDE$ be a regular pentagon and F be the intersection of diagonals AD and BE. The isosceles triangle AFE can be completed to a regular pentagon $AFEXY$, let us denote it p. There is another regular pentagon, q, the vertices of which are the intersections of all the five diagonals of $ABCDE$. Given that $AF = 1$, what is the largest distance between a vertex of p and a vertex of q?

Result. $\frac{3+\sqrt{5}}{2} \approx 2.61803$

Solution. Observe that the two pentagons, p and q, are homothetic with center F, therefore the line segment between X and Z, one of the two possible pairs of points of p and q maximizing the distance, passes through F.



Straightforward angle chasing reveals the following facts:

- $\angle AFE = 108°$, hence $AE = \frac{1}{2}(1+\sqrt{5})$ by the Law of Cosines,
- triangles AFZ, XFD and DFE are isosceles.

Therefore

$$XF = DF = DE = AE = \frac{1}{2}(1+\sqrt{5})$$

and $ZF = AF = 1$. The sought distance is $XF + ZF = \frac{1}{2}(3+\sqrt{5})$.

Problem 35. Consider all the triples (a, b, c) of prime numbers solving the equation

$$175a + 11ab + bc = abc.$$

What is the sum of all possible values of c in these solutions?

Result. 281

Solution. We can transform the equation to $a(bc - 11b - 175) = bc$. From this we see that either $a = b$ or $a = c$ since all the variables should be primes. In the first case, we get

$$ac - 11a - 175 = c \quad \iff \quad (a-1)(c-11) = 186,$$

giving only the solution $(2, 2, 197)$ in primes. The second case yields

$$ab - 11b - 175 = b \quad \iff \quad 175 = b(a-12),$$

and the other two solutions in primes are $(47, 5, 47)$ and $(37, 7, 37)$. Therefore the value sought is $197 + 47 + 37 = 281$.

Problem 36. Tom and Mary want to buy a house. They look for a perfect place, but their definitions of "perfect" differ. They found 10 offers and decided to try the following decision process: they both rank the houses randomly (a draw is not allowed) and if the top-three houses of Mary and Tom have exactly one house in common, they buy this house. What is the chance that this process succeeds?

Result. $\frac{21}{40}$

Solution. For any ranking of Mary, Tom needs to have exactly one house of Mary's top 3 in his top 3 and the remaining two of Mary's top 3 among his rankings 4 to 10. So for any ranking of Mary, Tom has a chance of

$$\frac{\binom{3}{1} \cdot \binom{7}{2}}{\binom{10}{3}} = \frac{21}{40}.$$

Problem 37. Let us define polynomials

$$p(x) = ax^{2021} + bx^{2020} + \cdots + ax^{2k-1} + bx^{2k-2} + \cdots + bx^2 + ax + b$$

and

$$q(x) = ax^2 + bx + a,$$

where a and b are positive real numbers. We know that $q(x)$ has precisely one real root. Find the sum of all real roots of $p(x)$.

Result. -2

Solution. Since the polynomial $p(x)$ can be factorized into $(ax+b)(x^{2020} + x^{2018} + \cdots + x^2 + 1)$ where the second factor is positive, the only real root of $p(x)$ is $x = \frac{-b}{a}$. Given that the quadratic polynomial $q(x)$ has only one real root, it must be a double root. Hence we get $b^2 - 4a^2 = 0$ for the discriminant, and since a, b are positive we can say $b = 2a$. Therefore, the only real root equals $x = \frac{-b}{a} = \frac{-2a}{a} = -2$.

Problem 38. Find the sum of all prime numbers p such that there exists some positive integer n such that the decimal expansion of $\frac{n}{p}$ has the shortest period of length 5.

Result. 312

Solution. We can assume that $n < p$ and that the decimal expansion of $\frac{n}{p}$ is 5-periodic starting right after the decimal point. Indeed, if $\frac{n}{p}$ is only *eventually* periodic, one can shift the decimal point by multiplying n by a suitable power of 10 and then, if $n \geq p$, we can replace it by $n' < p$ such that $n = kp + n'$: this only "erases" the part in front of the decimal point.

If $0.\overline{ABCDE}$ is the periodic decimal expansion of $\frac{n}{p}$ then $99999 \cdot \frac{n}{p} = 10^5 \cdot \frac{n}{p} - \frac{n}{p} = ABCDE$ is an integer. Since $n < p$ and p is a prime, it follows that $p \mid 99999$, that is $p \mid 3^2 \cdot 41 \cdot 271$. Both $\frac{1}{3}$ and $\frac{2}{3}$ have the shortest period 1, but $\frac{1}{41} = 0.\overline{02439}$ and $\frac{1}{271} = 0.\overline{00369}$ are of the desired type. Therefore, the result is $41 + 271 = 312$.

Problem 39. Four people are sitting in a room, each of them speaks exactly three of these five languages: Czech, German, English, Polish, and Hungarian. They don't speak any other language. We can see that in total there are 10000 ways to assign the languages to people. In how many of these scenarios can someone give a talk in a language that all of them understand?

Result. 5680

Solution. We will use the inclusion–exclusion principle to compute the result. Let us number the languages in some order by $1, \ldots, 5$. Denoting by A_i the set of assignments of the languages to the people such that all of them speak the i-th language, we are asked for $\left|\bigcup_{i=1}^{5} A_i\right|$, the size of the union of the sets. Firstly, $|A_i| = \binom{4}{2}^4 = 6^4$ since we can choose the two of the four remaining languages for all the four people independently. There are 5 possible choices of i. For any fixed $i \neq j$, we similarly obtain that $|A_i \cap A_j|$, the number of language assignments for which everybody speaks both the i-th and the j-th language equals 3^4. Clearly, there are $\binom{5}{2} = 10$ such pairs of indices $i \neq j$. Finally, there is exactly one assignment where all the people speak the same fixed three languages and there are $\binom{5}{3}$ such choices of three different languages (in our notation, we just observed that $|A_i \cap A_j \cap A_k| = 1$ for any of the 10 choices of pairwise different indices i, j, k). The aforementioned inclusion–exclusion principle then yields

$$\left|\bigcup_{i=1}^{5} A_i\right| = 5 \cdot 6^4 - 10 \cdot 3^4 + 10 = 6480 - 810 + 10 = 5680.$$

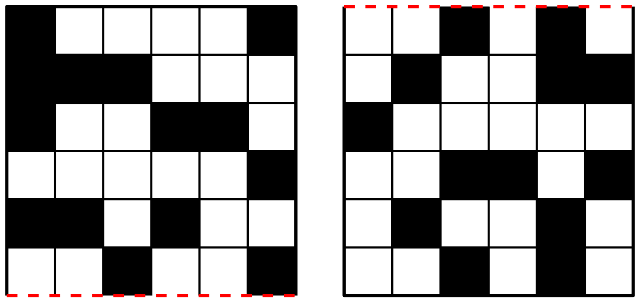
Problem 40. Julie writes down all the fractions whose numerators and denominators are positive integers smaller or equal to 100, erases any which are not in lowest terms and then lists the rest from least to greatest. In Julie's list, which fraction comes immediately before $\frac{2}{3}$?

Result. $\frac{65}{98}$

Solution. We should try fractions with denominators as large as possible, because if $\frac{a}{b}$ is less than $\frac{2}{3}$, then $\frac{a+2}{b+3}$ is greater than $\frac{a}{b}$ and still less than $\frac{2}{3}$. In particular, this means that we should try the denominators 98, 99, and 100. Thus, the possible solutions are $\frac{65}{98}$, $\frac{65}{99}$, and $\frac{66}{100} = \frac{33}{50}$. Comparing these fractions, we find that

$$\frac{65}{99} < \frac{33}{50} < \frac{65}{98} < \frac{2}{3}.$$

Problem 41. Twenty-three black unit cubes are placed in a $6 \times 6 \times 6$ grid. The figure shows what the resulting object looks like from above (the left square) and from the front (the right square). A white square means that there is no black cube in the respective column. The common edge of the two corresponding faces of the grid is marked in red. Determine the surface area of the black object.



Result. 130

Solution. The perimeter equals the sum of the perimeters of the black unit cubes minus twice the number of faces shared by a pair of black cubes. Such a pair can be oriented in three directions, let us call them "up-down", "front-back" and "left-right". If the "left-right" case occurs, it must appear also as a pair of two black squares sharing a vertical side separating the same two columns in both of the projections. Checking column by column, we easily check that this never happens. The "front-back" cases must affect the left square – it happens twice in the first column. As the first column of the front overview contains a single black cell, the positions of the black cubes in the leftmost layer of the big cube are uniquely determined and there are indeed two vertical faces shared by two black cubes. The last case "up-down" can be treated similarly revealing that it contributes two shared faces (due to the fifth columns of the projections). We conclude that the desired surface area is $6 \cdot 23 - 2 \cdot 4 = 130$.

Problem 42. A *dividing decomposition* of the positive integer N is a sequence of positive integers $d_1, d_2, \ldots, d_k$ such that $k \geq 1$, $d_1 \neq 1$, the divisibility conditions $d_1 \mid d_2 \mid d_3 \mid \cdots \mid d_k \mid N$ hold, and $d_1 \cdot d_2 \cdots d_k = N$. We will call the number d_k the *leader* of the decomposition. What is the arithmetic mean of the leaders among all dividing decompositions of 720?

Result. 204

Solution. We have $720 = 2^4 \cdot 3^2 \cdot 5$. The exponents of any prime p in $d_1, d_2, \ldots, d_k$ form a non-decreasing sequence, which sums up to the exponent of p in 720. For the prime 2, this sequence might be $(1,1,1,1)$, $(1,1,2)$, $(2,2)$, $(1,3)$, (4) or any of these preceded by some number of zeros. For 3, the sequences end $(1,1)$ or (2). For 5, there is just (1). Any combination of these sequences gives us a dividing decomposition. Therefore, there are 10 dividing decompositions of 720 and the arithmetic mean of their leaders is $\frac{2+4+4+8+16}{5} \cdot \frac{3+9}{2} \cdot 5 = 204$.

Problem 43. The *Scrabboj* game set consists of a 5×1 board and a bag of distinguishable tiles. On each tile exactly one letter out of N, A, B, O, J is written. How many different sets of Scrabboj are there for which the total number of ways to compose the word *NABOJ* is equal to 1440?

Result. 9450

Solution. Denote by n, a, b, o, j the numbers of tiles with letters N, A, B, O, J, respectively. We are looking for the number of 5-tuples (n, a, b, o, j) with

$$n \cdot a \cdot b \cdot o \cdot j = 1440 = 2^5 \cdot 3^2 \cdot 5.$$

Exponents of each prime can be independently distributed among the numbers n, a, b, o, j and different distributions yield different 5-tuples. For example for the prime 2, we need to divide 5 objects into 5 boxes. This can be done in $\binom{9}{4}$ ways, indeed we can choose which of 9 things are objects and which ones are dividers between the boxes. Similarly, for 3 there are $\binom{6}{4}$ ways and for 5 there are $\binom{5}{4}$ ways. Therefore there are

$$\binom{9}{4} \cdot \binom{6}{4} \cdot \binom{5}{4} = 126 \cdot 15 \cdot 5 = 9450$$

different *Scrabboj* sets.

Problem 44. Find the largest positive integer n such that $4^{2021} + 4^n + 4^{3500}$ is a perfect square.

Result. 4978

Solution. Assume that the largest such integer n is at least 2021. Then, after dividing $4^{2021}+4^n+4^{3500}$ by $4^{2021}=\left(2^{2021}\right)^2$, we get another square

$$1+(2^m)^2+2^{2958},$$

where $m=n-2021$. Moreover, it is a square of a number larger than 2^m: let us write

$$(2^m)^2+2^{2958}+1=(2^m+x)^2$$

for some positive integer x. Then $x\cdot 2^{m+1}+x^2=2^{2958}+1$. The left-hand side is increasing in both m and x while the right-hand side is a constant, so the solution with the largest m will have the smallest possible x. If we try to take $x=1$, then $m=2957$ solves this equation and by reverting the previous argument we see that $n=m+2021=4978$ works. This justifies our initial assumption that the largest admissible n is at least 2021 and we conclude that 4978 is the desired maximal value.

Problem 45. How many coefficients of the polynomial

$$P(x)=\prod_{i=2}^{2021}(x^i+(-1)^i i)=(x^2+2)(x^3-3)(x^4+4)\cdots(x^{2021}-2021)$$

are positive (strictly bigger than zero)?

Result. 1021616

Solution. Consider the polynomial

$$Q(x)=P(-x)=(x^2+2)(-x^3-3)(x^4+4)\cdots(-x^{2021}-2021)=(-1)^{1010}(x^2+2)(x^3+3)(x^4+4)\cdots(x^{2021}+2021)$$

and notice that all its nonzero coefficients are positive. We claim that these are exactly the ones corresponding to the powers x^k for k between the minimum possible one, i.e. 0, and the maximum possible one, i.e.

$$S:=2+\cdots+2021=2043230,$$

except for exactly the two numbers 1 and $S-1$. When we imagine the product of the 2020 factors defining Q expanded, it is clear that there will be no linear terms and it is also easy to see that the same holds for x^{S-1}: if we choose the power of x from every bracket, we obtain x^S, otherwise the exponent is at most $S-2$.

Now we prove that every other exponent from the range above is present with a positive coefficient or, equivalently, that the smallest number m larger than 1 that cannot be written as a sum of a subset of $\{2,3,\ldots,2021\}$ equals $S-1$. We claim that

$$m-1=k+(k+1)+\cdots+2021$$

for some $k\in\{2,3,\ldots,2021\}$. Indeed, clearly $m\geq 3$ and hence it must be possible to write $m-1$ as a sum of some subset of $\{2,3,\ldots,2021\}$. Moreover, for every subset different from $\{k,k+1,\ldots,2021\}$ we can just increase one of the numbers in the sum by 1 and obtain a valid representation of m. Moreover, $k\leq 3$ as otherwise

$$2+(k-1)+(k+1)+\cdots+2021$$

is a way to express m. Hence $m=1+3+4+\cdots+2021=S-1$.

Therefore $Q(x)$ has $\frac{S}{2}+1$ positive coefficients at even powers of x and $\frac{S}{2}-2$ positive coefficients at odd powers of x. The original polynomial $P(x)$ has the signs at the odd coefficients flipped, and hence it has exactly $\frac{S}{2}+1=1021616$ positive coefficients.

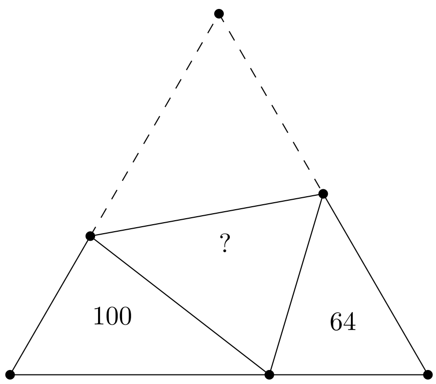
Problem 46. The *Cube City of Tomorrow* has a map that looks like a cubic grid $4\times4\times4$. Every point with integral coordinates is called a *crossroad* and every two crossroads of distance 1 are connected by a straight road. The crossroad in the middle of the city, $(2,2,2)$, is closed due to maintenance. David wants to go from the crossroad $(0,0,0)$ to the crossroad $(4,4,4)$ via the shortest possible path along roads. How many possible paths are there?

Result. 26550

Solution. Firstly, we will calculate how many shortest paths there are without condition. We have to go from $(0,0,0)$ to $(4,4,4)$. We have to take four roads in x direction, four in y direction and four in z direction, in any order. If we go back, then the path will not be the shortest. That is $\frac{12!}{4!\cdot4!\cdot4!}$ possible paths.

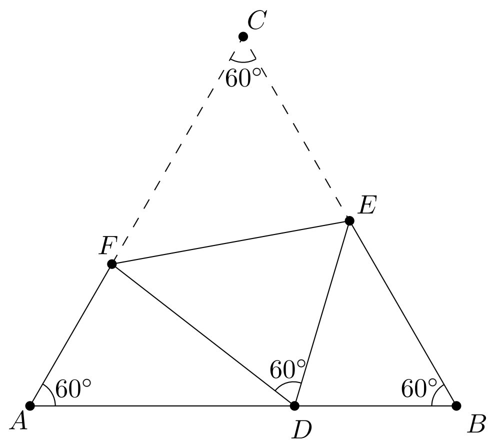
Now we have to subtract those paths that go through crossroad $(2,2,2)$. Due to symmetry we know that the number of paths from $(0,0,0)$ to $(2,2,2)$ is equal to the number of paths from $(2,2,2)$ to $(4,4,4)$ and that is equal to $\frac{6!}{2!\cdot2!\cdot2!}$. For any path from $(0,0,0)$ to $(2,2,2)$, there are $\frac{6!}{2!\cdot2!\cdot2!}$ possible continuations from $(2,2,2)$ to $(4,4,4)$. Hence the number of all roads through crossroad $(2,2,2)$ is $\frac{6!}{2!\cdot2!\cdot2!}\cdot\frac{6!}{2!\cdot2!\cdot2!}=\frac{6!\cdot6!}{2^6}$. The total number of paths is thus $\frac{12!}{4!^3}-\frac{6!^2}{2^6}=26550$.

Problem 47. An equilateral triangle is folded in such a way that one vertex hits exactly the opposite side and the areas of the two newly formed non overlapping triangles are 100 and 64 as in the picture. Find the area of the overlapping triangle.

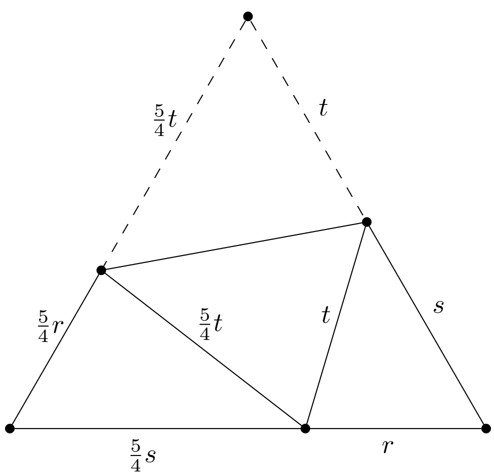


Result. 98

Solution. The relevant points are labeled as in the picture.



Since triangle ABC is equilateral, we get $\angle BDE + \angle DEB = 120° = \angle BDE + \angle FDA$, i.e. $\angle DEB = \angle FDA$ and therefore $ADF \sim BED$. Since the areas of these triangles have the ratio $100 : 64$, the respective sides have the ratio $5 : 4$. If we set $r = DB$, $s = BE$, $t = ED$, we get the lengths as labeled in the following picture.



We can derive the following equations using a for the side length of the equilateral triangle:

$$a = s + t, \tag{2}$$

$$a = \frac{5}{4}(r + t), \tag{3}$$

$$a = \frac{5}{4}s + r, \tag{4}$$

$$64 = \frac{1}{2}rs \cdot \sin 60° = rs \cdot \frac{\sqrt{3}}{4}. \tag{5}$$

The linear equations (2)–(4) yield $r = \frac{a}{3}$ and $s = \frac{8a}{15}$. Inserting these values in the equation (5) then gives

$$1440 = a^2\sqrt{3} = 4S$$

where S is the area of the equilateral triangle. It follows that the desired area A satisfies $100 + 64 + 2A = 360$ and hence $A = 98$.

Problem 48. Flip a fair coin repeatedly until the sequence heads-tails-heads occurs. What is the probability that the sequence tails-heads-tails-heads has not yet occurred?

Result. $\frac{5}{8}$

Solution. Let us denote by E the event that sequence "heads-tails-heads", or simply HTH, occurs before $THTH$ and by $\mathrm{P}(E)$ its probability. For a fixed finite sequence s of heads and tails denote by $\mathrm{P}(E \mid s)$ the probability that E occurs in a sequence starting by s and continuing randomly. Set $x = \mathrm{P}(E \mid H)$ and $y = \mathrm{P}(E \mid T)$. Since our coin is fair, moving one or two steps ahead (we always write the new outcome to the right end of the current sequence) we compute

$$x = \frac{1}{2}\mathrm{P}(E \mid HH) + \frac{1}{4}\mathrm{P}(E \mid HTT) + \frac{1}{4}\mathrm{P}(E \mid HTH) \tag{6}$$

and analogously by moving up to three steps ahead we obtain

$$y = \frac{1}{2}\mathrm{P}(E \mid TT) + \frac{1}{4}\mathrm{P}(E \mid THH) + \frac{1}{8}\mathrm{P}(E \mid THTT) + \frac{1}{8}\mathrm{P}(THTH). \tag{7}$$

Since both HTH and $THTH$ are alternating sequences, we have

$$\begin{aligned} x &= \mathrm{P}(E \mid H) = \mathrm{P}(E \mid HH) = \mathrm{P}(E \mid THH), \\ y &= \mathrm{P}(E \mid T) = \mathrm{P}(E \mid TT) = \mathrm{P}(E \mid HTT) = \mathrm{P}(E \mid THTT). \end{aligned}$$

Moreover, as $\mathrm{P}(E \mid HTH) = 1$ and $\mathrm{P}(E \mid THTH) = 0$, equations (6) and (7) read as

$$\begin{aligned} x &= \frac{x}{2} + \frac{y}{4} + \frac{1}{4}, \\ y &= \frac{y}{2} + \frac{x}{4} + \frac{y}{8}. \end{aligned}$$

That gives us $x = \frac{3}{4}$ and $y = \frac{1}{2}$. The desired probability is then $\mathrm{P}(E) = \frac{1}{2}\mathrm{P}(E \mid H) + \frac{1}{2}\mathrm{P}(E \mid T) = \frac{x+y}{2} = \frac{5}{8}$.

Problem 49. Find the smallest positive real number x with the following property: There exists at least one triple of positive real numbers (s, t, u) such that

$$\begin{aligned} s^2 - st + t^2 &= 12, \\ t^2 - tu + u^2 &= x, \end{aligned}$$

and no two triples with this property can differ in the last coordinate only.

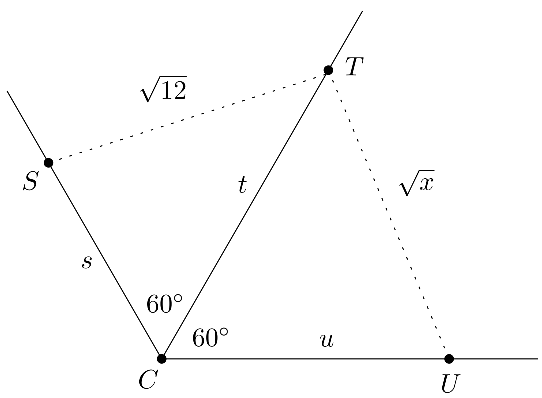
Result. 16

Solution. Consider points S, T, U, and C in the plane such that $CS = s$, $CT = t$, $CU = u$ and

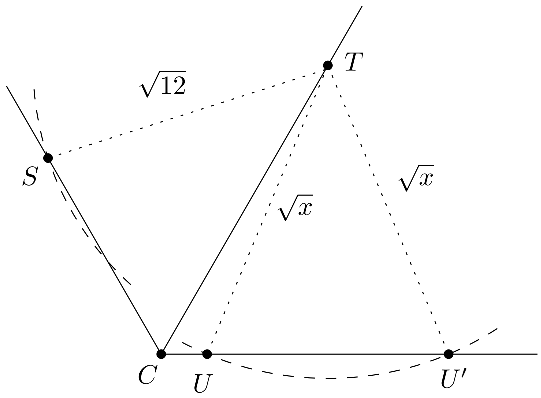
$$\angle SCT = \angle TCU = 60^\circ$$

as on the picture below. By the Law of Cosines ($\cos 60^\circ = \frac{1}{2}$), the equations in the statement now imply $ST^2 = 12$ and $TU^2 = x$. Since the distance from T to line SC is at most $\sqrt{12}$ with the equality if and only if $\angle TSC = 90^\circ$, we obtain

$$t \le \frac{\sqrt{12}}{\sin(60^\circ)} = 4.$$



In the following arguments we fix point C and the three rays emanating from it and move with any of the points S, T, U only within the respective ray. If $\sqrt{x} < 4$, it is possible to place the segment TU in the angle UCT (i.e. to find admissible values t, u) so that $CU < CT \leq 4$ and $\angle CUT \neq 90°$ (just put U very close to C). It follows from the first inequality and the angle condition that the circle centered at T with radius $\sqrt{x}$ intersects the ray CU in two different points U and U' (see the second figure) which contradicts the given condition concerning the uniqueness of u. The second inequality implies that the circle centered at T with radius $\sqrt{12}$ intersects the ray CS in at least one point S. These two facts imply that $x \geq 4$.



For $\sqrt{x} = 4$ (resp. for any fixed $\sqrt{x} \geq 4$) and any $0 < t \leq 4$ there is only one such intersection U and hence the mentioned condition is satisfied. Also, analogously as above, the circle centered at T with radius $\sqrt{12}$ intersects the ray CS in at least one point S and hence the resulting triple $(s, t, u) = (CS, CT, CU)$ solves the given system of equations for our x. It follows that the smallest x satisfying the given conditions is $4^2 = 16$.

Note. Alternatively, these geometric arguments can be replaced by using the characterization of the number of solutions of a quadratic equation by its discriminant.

Problem 50. For how many $x \in \{1, 2, 3, \ldots, 2020\}$ is it possible that Marek summed 2020 non-negative consecutive integers, Michal summed $2020 + x$ non-negative consecutive integers and they got the same result?

Result. 1262

Solution. Let n denote the first term of Marek's sum and m the first term of Michal's sum. Then

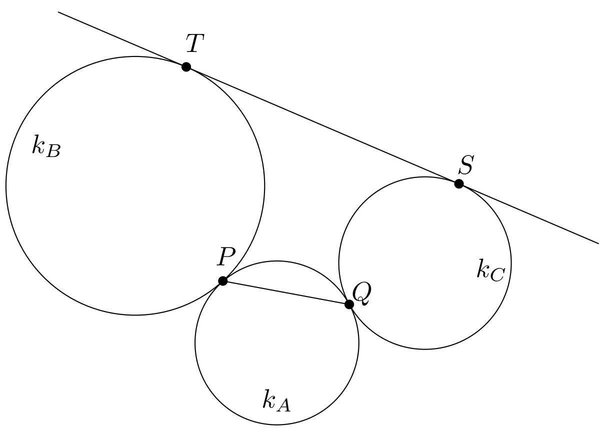
$$\begin{aligned} 2020n + \frac{2019 \cdot 2020}{2} &= (2020 + x)m + \frac{(2019 + x)(2020 + x)}{2}, \\ 2020(n - m) &= x\frac{2m + 2019 + 2020 + x}{2}. \end{aligned} \tag{8}$$

Since the left-hand side of (8) is divisible by four, so must be the right-hand side and hence $4 \mid x\left(m + \frac{x-1}{2}\right)$. It follows that either x is odd (so that the bracket can be an integer divisible by four) or $8 \mid x$ (if x is even, then $x - 1$ is odd and we lose one power of 2 at the fraction).

One can directly check that numbers $x = 2k + 1$ for $k \in \{0, 1, \ldots, 1009\}$, $m = 2020 - k$ and $n = 2020 + 3k + 2$ satisfy the equation (8). For $x = 8k$, $k \in \{1, 2, \ldots, 252\}$ one can similarly check that (8) holds for $m = 1263 - 4k$ and $n = 1263 + 9k$ (note that m and n are positive for all considered values of k).

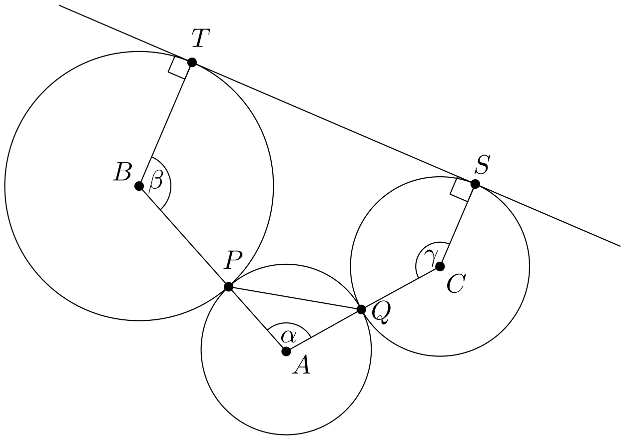
In summary, we found $1010 + 252 = 1262$ possible values of x and proved that there are no more.

Problem 51. Circles k_B and k_C touch circle k_A in points P and Q, respectively. Find the radius r_A of the circle k_A, if the radii of k_B and k_C are $r_B = 5$ and $r_C = 3$, respectively, $PQ = 6$ and the outer tangent segment $TS = 12$.



Result. $\frac{4+\sqrt{61}}{3} \approx 3.93675$

Solution. Let A, B, C denote the centres of the circles. Furthermore, let $\alpha = \angle QAP$, $\beta = \angle PBT$, $\gamma = \angle SCQ$. Since $BT \perp TS$ and $CS \perp TS$ we get $\alpha + \beta + \gamma = 360°$ due to the sum of interior angles in pentagon $TBACS$.



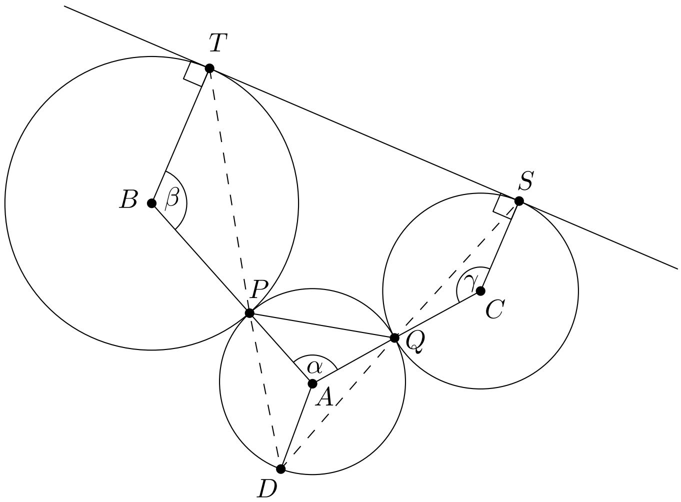
Since TS is a tangent line to the circles k_B and k_C, we get $\angle PTS = \frac{1}{2}\beta$ and $\angle TSQ = \frac{1}{2}\gamma$. Due to

$$\angle SQP = 180° - (90° - \frac{1}{2}\alpha) - (90° - \frac{1}{2}\gamma) = \frac{1}{2}(\alpha + \gamma),$$

we conclude that $\angle PTS + \angle SQP = 180°$ and therefore quadrilateral $PQST$ is cyclic. Let D denote the point of intersection of the lines TP and SQ. Since $PQST$ is cyclic, the triangles DST and DQP are similar, which leads to

$$\frac{PQ}{TS} = \frac{DP}{DS} = \frac{DQ}{DT}.$$

Using again $\angle PTS = \frac{1}{2}\beta$ and $\angle TSQ = \frac{1}{2}\gamma$, we get $\angle SDT = \angle QDP = \frac{1}{2}\alpha$, which means that D lies on circle k_A.



Therefore we have $\angle DPA = \angle TPB$ and the triangles APD and BPT are similar, too. By analogy, we get $\triangle ADQ \sim \triangle CSQ$. From these similarities we derive the relations

$$\frac{TP}{DP} = \frac{r_B}{r_A} \quad \text{and} \quad \frac{SQ}{DQ} = \frac{r_C}{r_A}$$

leading to

$$\frac{DT}{DP} = \frac{r_A + r_B}{r_A} \quad \text{and} \quad \frac{DS}{DQ} = \frac{r_A + r_C}{r_A}.$$

Inserting these results in above equation gives

$$\frac{TS^2}{PQ^2} = \frac{DS}{DP} \cdot \frac{DT}{DQ} = \frac{(r_A + r_B) \cdot (r_A + r_C)}{r_A^2}.$$

Now we can compute r_A by plugging in the values given and we obtain the quadratic equation

$$\frac{144}{36} \cdot r_A^2 = r_A^2 + 8r_A + 15 \quad \Longleftrightarrow \quad 3r_A^2 - 8r_A - 15 = 0.$$

The two solutions of this equation are given by

$$\frac{8 \pm \sqrt{64 + 12 \cdot 15}}{6} = \frac{4 \pm \sqrt{61}}{3}.$$

The only positive value from these solutions is $\frac{4+\sqrt{61}}{3} \approx 3.93675$.